MAT 257, Handout 3: week of September 26-30, 2011.

I. Assignment.

Note that this is *not* to be handed in. The homework that will be collected appears in Section III below, on page 2.

Over the next week. Read pages 15-25 of Spivak, and work a large sample of the problems on pages 17-19 and 22-25.

As usual, make sure to solve *in detail* all of the starred exercises, which in this week is just exercise 2-1 on page 17.

II. Summary.

1. Let X be a metric space with metric $d : X \times X \to [0, \infty)$. We noted that a subset U of X is open if and only if

$$\text{for every } x \in U, \text{ there exists } r > 0 \text{ such that } \{y \in X : d(y, x) < r\} \subset U. \tag{1}$$

(Compare (1) to the slightly different definition we gave earlier, which you can find in Handout 2. If the equivalence between that definition and (1) is not clear to you, then write out the proof in detail.) In practice, it is often more convenient to work with (1) than with the earlier definition.

2. Given a metric space X with a metric denoted $d_X : X \times X \to [0, \infty)$, and a subset $A \subset X$, we can make A into a metric space, with metric denoted $d_A : A \times A \to [0, \infty)$, defined by

$$d_A(x, y) = d_X(x, y) \qquad \text{for } x \text{ and } y \in A.$$

Thus, d_A is just the restriction to $A \times A$ of d_X. (We sometimes call d_A the *induced metric*. It is easy to check that it is in fact a metric.) Having made A into a metric space by defining d_A, we can define open sets in A, just as we would do in any other metric space, using (1) (with d replaced by d_A), or an equivalent definition of "open" if we prefer,

We checked that a set $U \subset A$ is open in A if and only if we can write $U = A \cap V$ for some $V \subset X$ such that V is open in X.

Note that in this situation, a subset U of A can be open in A and not open in X. This is the case for example if $U = A = [0, 1] \subset X = \mathbb{R}$ (with the standard metric).

We sometimes say "relatively open in A" to mean the same thing as "open in A". We also sometimes say simply "relatively open" to mean "open in A", when it is clear from the context which set $A \subset X$ we have in mind.

Later on, we will normally consider $X = \mathbb{R}^n$ and A a subset of $\mathbb{R}^n$. In this situation, we will *never* use the notation d_A and d_X. Rather, we will always write simply $|x - y|$ to denote the Euclidean distance between two points (whether we regard them as points in A or in X.)

3. Suppose that X and Y are metric spaces, with metrics d_X and d_Y respectively. We say that a function $f : X \to Y$ is *continuous* at a point $x \in X$ if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ such that } d_Y(f(x), f(x')) < \varepsilon \text{ for every } x' \in X \text{ such that } d_X(x', x) < \delta. \tag{2}$$

If f is continuous at every point in X, then we simply say that f is continuous.

We checked that

$$f : X \to Y \text{ is continuous } \iff \text{ for every open } U \subset Y,\ f^{-1}(U) \text{ is open in } X. \tag{3}$$

An advantage of defining the induced metric as we did above is that, having done so, (3) remains true with only the obvious modifications (replacing X by A throughout) for a function $f : A \to Y$, when A is a subset of a metric space X. (Compare the different approach in Spivak, in particular Theorem 1-8 and the discussion that precedes it, where the "inverse image" characterization of continuity looks different depending on whether the domain is $\mathbb{R}^n$ or a proper subset of $\mathbb{R}^n$.)

4. Compact sets in metric spaces enjoy the following good properties:

- if X and Y are metric spaces, K is a compact subset of X, and $f : K \to Y$ is continuous, then $f(K)$ is a compact subset of Y. (Essentially Theorem 1-9 in Spivak.)
- if K is a compact subset of a metric space X, and if $f : K \to \mathbb{R}$ is a real-valued function, then f attains its supremum and infimum. That is, there exist points x^* and x_* in K such that
$$f(x^*) = \sup_{x\in K} f(x), \qquad f(x_*) = \inf_{x\in K} f(x).$$
(Essentially exercise 1-29 in Spivak, which was discussed in class.)
- if X and Y are metric spaces, K is a compact subset of a metric space X, and $f : K \to Y$ is continuous, then f is uniformly continuous, which means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any x and y in K satisfying $d_X(x, y) < \delta$, we have the inequality $d_Y(f(x), f(y)) < \varepsilon$. (As we well recall from MAT157, the point is that "the same δ works for every x and y".)
  See exercise (2) below.
- Every sequence in a compact set has a convergent subsequence. (One approach to this, using the notion of "totally bounded", is hinted at in Handout 2.) Incidentally, this property of compact sets can be used to provide an alternate proof of the fact that a continuous function on a compact set attains its infimum and supremum.

5. We did not discuss material about the concept of oscillation and its relation to continuity (from page 13 of Spivak.) We will return to this when we need it, which will be later in the fall.

III. Hand in on October 7.

(1) Exercise 1-21 in Spivak.

(2) Prove that if X and Y are metric spaces, K is a compact subset of X, and $f : K \to Y$ is continuous, then f is uniformly continuous.

Hint: You may as well start by fixing $\varepsilon > 0$; then your goal is to prove the existence of a suitable $\delta > 0$, as in the definition of uniform continuity. As usual, to exploit the compactness of K, you can construct a specific open covering of K that somehow encodes some useful information. In this case, this might be information about which points are close enough together that their images are within distance ε of each other.

(3) A subset A of a metric space X is *disconnected* if there exist relatively open sets $A_1, A_2 \subset A$ such that $A_1 \cup A_2 = A, \quad A_1 \cap A_2 = \emptyset$.

A set is *connected* if it is not disconnected.

(a) Prove that $\{x \in \mathcal{Q} : 0 \le x \le 1\}$ is disconnected, where $\mathcal{Q}$ denotes the rational numbers.

(b) Prove that a closed interval $[a, b] \subset \mathbb{R}$ is connected.

Discussion: if we knew the result from part (b) above, we could use it to prove the Heine-Borel theorem (compactness of a closed interval) as follows:

- Start as before: given an open cover $\mathcal{O}$ of $[a, b]$, we define
$$A := \{x \in [a, b] : [a, x]\ [a, x] \text{ can be covered by finitely many sets from } \mathcal{O}\}.$$
- Then verify that A is both open and closed in $[a, b]$.
- Finally, use the connectedness of $[a, b]$ to conclude that $A = [a, b]$.

This proof, once the details are filled in, would probably be a little harder than the one we gave, though the underlying idea is very similar.

(4) Exercise 2-4 in Spivak.