MAT 257, Handout 4: October 3-7, 2011.

I. Remarks about homework 2.

1. One of the homework problems due this Friday asks you to prove that every continuous function on a compact set is uniformly continuous.

The hint I provided for the problem is actually not very useful, so here is a bit more discussion.

First, the easiest way to solve the problem is probably not to start from the definition of compactness, but rather to exploit other facts that we (now) know about compactness, such as the fact that every sequence in a compact set K has a subsequence that converges to a limit in K. This was discussed in last week's tutorial, and for completeness is proved below (see Lemma 1). So feel free to use fact in your solution of the exercise.

Other proofs I know of "*continuity + compactness imply uniform continuity*" all use arguments similar to those in the proof of Lemma 1 below. For example, if you follow my earlier hint, you may find yourself constructing a finite collection of open balls $\{B_1, \ldots, B_L\}$ in K with the property that for each ball B_i, the set $\{f(x) : x \in B_i\}$ has diameter less than ϵ. To conclude from this that f is uniformly continuous, it would suffice to show that there exists some $\delta > 0$ such that, for every $x \in K$, the ball of radius δ centered at x is a subset of at least one of the B_i, $i = 1, \ldots, L$. If you try to prove this directly using the definition of compactness, you will probably find that you go through arguments like those in Lemma 1 below.

2. There is a typo in the statement of Problem 3. It should say:

A subset A of a metric space X is *disconnected* if there exist nonempty relatively open sets $A_1, A_2 \subset A$ such that $A_1 \cup A_2 = A, \quad A_1 \cap A_2 = \emptyset$.

A set is *connected* if it is not disconnected.

(The problem as originally stated failed to mention that the sets A_1 and A_2 must both be nonempty.)

II. Assignment.

Over the next week. Continue reading Chapter 2 of Spivak, through the end of page 32, and continue to work exercises (mostly not to be handed in) from Chapter 2.

Here are some remarks about the exercises.

- 2-1 is easy but important.
- In 2-2, the last sentence should be replaced by: "If $f(x, y) = g(x)$ for all x, y, and g is differentiable, show that f is differentiable, and express $f'(a, b)$ in terms of g'.
- You should be able to do 2-3 pretty quickly, 2-4 is assigned for the homework, and 2-5 through 2-7 are all useful practice.
- 2-8 is a special case of a Theorem 2-3(3), but you should make sure that you can do it.
- 2-9 is really a question about single-variable calculus.
- It is arguably worth doing one or two parts of problems 2-10 and/or 2-11. The point here is to compute derivatives using only basic properties of the derivative (eg chain rule, product rule, etc, but without partial derivatives) as is done in an example on the bottom of page 22. If nothing else, it will help you appreciate the theorems relating the derivative Df and partial derivatives. (Note that I went over problem 2-10(a) in the lecture.)
- 2-12 and 2-13 establish generalizations of Theorem 2-3(5). I particularly recommend problem 2-13, parts (a)-(c). (Part (d) is a question about single-variable calculus.)
- 2-14 and 2-15 are useful and important results (though I don't know if we will use them in this class.)
- 2-16 is also a good exercise, though the hint should make it quite straightforward.

- Concerning problems 2-17 through 2-20: your goal should be to practice enough with computing partial derivatives that you can do it as easily as you can compute an ordinary derivative of a function of a single variable. So do enough of these exercises (and more like them if necessary) to reach this level.
- Concerning problem 2-21: Part (b) could be stated more clearly as: "How should f be *redefined* so that $D_1 f(x,y) = g_1(x,y)$?" Parts (a) and (b) seem to give a hint for part (c), and the hint actually works, but it is not very clear *why* it works......
- 2-22 is a basic result. Please write out the solution carefully. (This is may appear on the next homework in any case.)
- 2-23 is good practice, and 2-24 is strongly recommended as illustrating a basic phemonenon one must be aware of.
- 2-25 through 2-27 are relevant for topics we will discuss later, and if you have had enough at this point, maybe it is best to return to them later, when they are needed.
- Problem 2-28 is useful practice.
- Problem 2-29 introduces the important concept of the *directional derivative* (which I will probably also mention in class), and problems 2-30 and 2-31 illustrate some possible pathologies. The phenomenon in 2-30, while one must be aware of it, does not occur of f is differentiable, as you have shown in Problem 2-29 (c), and is rarely encountered in practice.
- Exercise 2-32 shows that the converse to Theorem 2-8 is not true, and exercise 2-33 refines Theorem 2-8 a little. These are both worth doing, for a better understanding of a basic result.
- Exercises 2-34 is generally useful, and exercise 2-35 is used in establishing certain basic lemmas about differential geometry.

III. Other discussion.

Here is the proof promised above (and discussed in the tutorial last week.)

Lemma 1. *Assume that K is a compact subset of a metric space X, and that (x_n) is a sequence of points in K. Then there exists some $x \in K$ and a subsequence (x_{n_l}) such that $x_{n_l} \to x$ as $l \to \infty$.*

Proof. Suppose toward a contradiction that the conclusion fails, so that there exists a sequence (x_n) in K with no convergent subsequence.

Step 1. We first claim that for every $x \in K$, there exists some $r_x > 0$ such that the open ball of radius r_x about x contains only finitely many points of the sequence (x_n).

To prove this, suppose that some $x \in K$ does not have this property, so that every ball centered at x contains infinitely many terms of the sequence. Then we can certainly choose some n_1 such that x_{n_1} belongs to the ball of radius 1 about x, and proceeding inductively, and using our assumption about x, we can choose $n_{l+1} > n_l$ such that $x_{n_{l+1}}$ belongs to the ball of radius $2^{-(l+1)}$ about x. Then it follows (from the definition of convergence!) that the subsequence (x_{n_l}) chosen in this way converges to x, which contradicts our assumption that (x_n) has no convergent subsequences. This proves our claim

Step 2. Now consider the collection of open sets $\mathcal{O} = \{U_x\}_{x \in K}$, where for every $x \in K$, we define U_x to be the open ball with center x and radius r_x. That is, we define

$$U_x := \{y \in X : d(y,x) < r_x\}.$$

Every U_x is open and every $x \in K$ is contained in at least one ball (the ball U_x whose center is x itself) so $\mathcal{O}$ is an open cover of K. Thus there exists a finite subcover $U_{y_1}, \ldots, U_{y_M}$, for certain points $y_1, \ldots, y_M \in K$.

Step 3. Since there are infinitely many terms in the sequence (x_n) and only finitely many sets U_{y_i}, at least one of these sets must contain infinitely many points. But this is impossible, in view of the construction of U_{y_i}. Hence we have arrived at the desired contradiction. □