MAT 257, Handout 7: October 24-28, 2011.

Assignment.

Solve the exercises on this handout as well as exercise 2-41 in Spivak. (Not to be handed in).

Also, read pages 40-43 of Spivak, if you have not already done so, as well as the definition of a manifold on p. 109 of Spivak. Note that his definition differs slightly from the one given in the lecture, where we said that a diffeomorphism is a continuously differentiable bijection with a continuously differentiable inverse. One could be more precise by defining a C^k diffeomorphism as a bijection that is k times continuously differentiable, with its inverse also k times continuously differentiable. Then one defines a C^k manifold in the obvious way (i.e., replacing "diffeomorphism" by "C^k diffeomorphism" in the definition.) With this language, the definition I gave was really that of a C^1 manifold.

Test 1 Announcement

The first Term Test will take place **Thursday Nov 3, 4:10-6pm, EX 320**

Overall, the level of difficulty will be similar to that in previous exams (to be posted online shortly.) I hope to have a number of routine problems, amounting to about half the total marks, with the other problems being somewhat harder. The routine problems will include at least one in which you actually have to compute something like a derivative or a Jacobian matrix, and they may also include things like asking you to state a definition or a basic theorem.

I will try to set the exam so that students have enough time to attempt all problems. If there are hard questions, I will try to make them hard questions that can be solved with a relatively small amount of writing, once you have the right idea.

The exam will cover the first two chapters of Spivak, the lectures up to and including October 24, and the handouts and assignments, up to and including the exercises on this handout. Specifically:

1. Basic topology of metric spaces

- *Definitions*: Norms, inner products, and distance functions; continuity and convergence; interior, boundary and closure; open, closed, compact, and connected subsets of metric spaces; relatively open sets.
- *Theorems*: Cauchy-Schwarz inequality; Heine-Borel theorem; characterization of compact sets in $\mathbb{R}^n$; the continuous image of a compact set is again compact; minima and maxima of continuous functions on compact sets; all norms on a finite-dimensional vector space are equivalent; a compact subset of a metric space is sequentially compact (i.e., every sequence has a convergent subsequence), and a continuous function on a compact set is uniformly continuous.

2. Differentiation

- *Definitions*: Abstract definition of the derivative; partial derivatives and the Jacobian matrix; directional derivatives; continuously differentiable functions; manifolds in $\mathbb{R}^n$; second derivative.
- *Theorems*: The chain rule and other basic facts; a function with continuous partial derivatives is differentiable; symmetry of the second derivative; the Inverse Function Theorem and the Implicit Function Theorem.

Not included this time: oscillation of a real-valued function, higher order derivatives.

Should you memorize proofs of theorems? It is very hard to memorize all proofs of all theorems. In the long run, it is much more efficient, as well as useful and interesting, to first try to understand the proofs, and internalize the methods of proof, as well as possible; then to remember just an outline of the proof, or some key idea; roughly speaking, the minimum you would need to allow yourself to reconstruct the proof out of your base of general knowledge/understanding.

Some exercises

1. Prove the following:

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function in an open rectangle U, and if $\|Df(x)\| \le M$ at every $x \in U$, then

$$|f(x) - f(y)| \le M|x - y| \quad \text{for all } x, y \in U.$$

(Here $\|Df\|$ denotes the operator norm of Df.)

(This should be compared to Lemma 2-10 in Spivak, a similar result with a less natural statement and proof.)

2. Let $X := \{n \times n \text{ matrices } A : \det A \ne 0\}$ and consider the function $Inv : X \to X$ defined by

$$Inv(A) = A^{-1}.$$

a. Prove that $D(Inv)(A)(H) = -A^{-1}HA^{-1}$. (It may help to consider first the case $A = I$.)

b. Give a second proof, without relying on explicitly finding $D(Inv)(A)$, that Inv is a differentiable function at every point of X. (You will need to remember a little linear algebra for this.)

c. Can you show that Inv is twice differentiable on X? what about k times differentiable for arbitrary positive integer k? Which of the above proofs generalizes more easily?

3. for any $\alpha \in \mathbb{R}$, a function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ is said to be *homogeneous of degree* α if

$$f(tx) = t^\alpha f(x)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and $t > 0$. (If $\alpha > 0$ then the domain of f may be extended to all of $\mathbb{R}^n$, with $f(0) = 0$).

a. Prove that if f is everywhere differentiable, then f is homogeneous of degree α if and only if *Euler's formula*

$$\alpha f(x) = \sum_{i=1}^{n} x^i D_i f(x)$$

holds for all $x = (x^1, \ldots, x^n) \in \mathbb{R}^n \setminus \{0\}$. *Hint*: fix x and consider $g(t) = t^{-\alpha} f(tx)$.

b. Prove that if f is everywhere differentiable and homogeneous of degree α, then $D_j f^i$ is homogeneous of degree $\alpha - 1$.

4. Let $M^{n\times n}$ denote the space of $n \times n$ matrices, and let $O(n) := \{Q \in M^{n\times n} : Q^T Q = I\}$.

Let $S^{n\times n} := \{A \in M^{n\times n} : A = A^T\}$ denote the space of *symmetric* $n \times n$ matrices. Note that S is a vector space of dimension $\frac{1}{2}n(n+1)$.

a. Define $f : M^{n\times n} \to S^{n\times n}$ by $f(A) = A^T A$. Find the derivative $Df(I)$, and determine its range and its nullspace.

b. Use the fact that $f(QA) = A^T F(Q) A$ to find $Df(Q)$ at an arbitrary $Q \in O(n)$. Compute its range and its nullspace.

c. Use the Implicit Function Theorem to show that $O(n)$ is a manifold in $\mathbb{R}^{n^2}$. What is its dimension?