# MAT 257, Handout 9: November 7-11, 2011.

**Assignment.** Read pages 46-52 of Spivak and work the exercises on pages 49 and 52. In particular, we went over pages 46-49 quite lightly in the lecture, so make sure that you read this carefully.

Also, go back and read the discussion of *oscillation* on pages 12-13; this will be needed shortly, to prove Theorem 3-8, which characterizes integrable functions on a rectangle in $\mathbb{R}^n$.

About the exercises:

3-3 and 3-5 are important exercises that establishes basic properties (linearity and monotonicity) of the integral. They are both straightforward. Should you ever study Lebesgue integration, you will find that the identity $\int(f+g) = \int f + \int g$ is considerable harder to prove for Lebesgue integrals than for Riemann integrals.

3-6 is a basic fact that is used all the time, and it is also part of the homework assignment.

3-2 is closely related to Theorem 3-8, to be proved shortly, so doing this exercise will probably help you digest that basic result.

3-7 is also an interesting exercise, and also related to Theorem 3-8.

3-1 and 3-5 are both good practice with the definition of the integral.

I strongly recommend that you solve *all the exercises* on page 52.

### Some topics from the lectures

We noted that a second-order Taylor series of function $f : \mathbb{R}^n \to \mathbb{R}$ is particularly useful at a critical point of a function $f$. If $a$ is a critical point of $f$ and $f$ is twice differentiable at $a$, then

$$f(a+x) = f(a) + \frac{1}{2}\sum_{i,j=1}^{n} D_{i,j}f(a)x^i x^j + R_2(x)$$

where $\frac{R_2(x)}{|x|^2} \to 0$ as $x \to 0$. (Note the matrix $D_{i,j}f$ is sometimes called the "Hessian matrix.")

In particular, under the above conditions,

- if the matrix $D_{i,j}f(a)$ is positive definite (i.e. all eigenvalues are positive), then $a$ is a local minimum of $f$.
- if the matrix $D_{i,j}f(a)$ is negative definite, then $a$ is a local maximum of $f$.
- if $D_{i,j}f(a)$ has $k$ negative eigenvalues and $n-k$ positive eigenvalues, for $1 \le k \le n-1$, then $f$ has a sort of saddle point at $a$.
- Note that it 0 is an eigenvalue of $D_{i,j}f(a)$, then we can say much less. For example if all eigenvalues of $D_{i,j}f(a)$ are nonnegative (but some vanish), then we cannot conclude that $f$ has a local minimum at $a$.

The above facts lead to an algorithm for trying to find maxima or minima of a function $f$ on an open set $U$ (bearing in mind that $f$ may not attain its extreme values on an open set):

(1) first find all critical points of $f$ on $U$. This amounts to looking for solutions of the system $\nabla f = 0$, which is a system of $n$ nonlinear equations in $n$ unknowns.
(2) Then compute the matrix $D_{i,j}f$ at each critical point. If 0 is not an eigenvalue, then you can determine the "character" of the critical point from the above discussion.
(3) If 0 is an eigenvalue of the Hessian matrix at a critical point $a$, then you will have to use other means to determine the character of the critical point.

**Also**, the exercises below refer to the *geometric version of the implicit function theorem.* This was discussed at length in the lectures but has not yet explicitly appeared on any of the handouts, so we state it here:

**Theorem 1.** *Assume that Assume that $g : \mathbb{R}^n \to \mathbb{R}^k$ is a continuously differentiable function, and that $p \in \mathbb{R}^k$ is such that the Jacobian matrix $g'(x)$ has rank $k$ at every point in $g^{-1}(p)$.*

*Then $M := g^{-1}(p)$ is a $C^1$ manifold of dimension $n-k$, which means that for every $x \in M$, there exist open sets $U, V \subset \mathbb{R}^n$ and a $C^1$ diffeomorphism $h : U \to V$ (that is, a continuously differentiable bijection with continuously differentiable inverse) such that $x \in U$ and $h(U \cap M) = V \cap (\mathbb{R}^{n-k} \times \{0\})$.*

In the exercises below, you will prove

**Theorem 2.** *Assume that $g$ and $M$ are as in the above theorem, and assume also that $F : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Let $f$ denote the restriction of $F$ to $M$. If $f$ has a local maximum or minimum at a point $x \in M$, then there exist $\lambda_1, \ldots, \lambda_k$ such that*

$$D_j F(x) = \sum_{i=1}^{n} \lambda_i D_j g^i(x) \qquad \text{for } j = 1, \ldots, n.$$

The assumption that $f$ has a local maximum (say) at a point $x \in M$ means, of course, that there exists an open set $U \subset \mathbb{R}^n$ such that $x \in U$ and $f(x) \geq f(y)$ for all $y \in U \cap M$. Equivalently, there exists a relatively open subset $W$ of $M$ such that $x \in W$ and $f(x) \geq f(y)$ for all $y \in W$.

The numbers $\lambda_1, \ldots, \lambda_k$ are called *Lagrange multipliers.* The conclusion of the theorem can be written in a number of different, equivalent ways, including for example $\nabla F(x) = \sum_{i=1}^{k} \lambda_i \nabla g^i(x)$. If we write it in terms of gradients, then we can interpret the theorem as stating that at a local maximum or minimum of $f$, "the gradient of $F$ is orthogonal to $M$".

## Homework 4, due November 18

**1**. Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ with the following three properties:

- $f$ is everywhere differentiable.
- The restriction of $f$ to any line through the origin has a strict local minimum at 0.
- But $f$ does not have a local minimum at the origin.

**hints:** In fact there exist *polynomials of rather low degree* with all the above properties.

You might as well construct $f$ so that $f(0) = 0$. Then the third requirement is satisfied if every neighbourhood of 0 contains points at which $f$ is negative, and for the second requirement to be satisfied, it suffices to arrange that for any line through the origin, there is an interval about the origin in which $f$ is positive (except at the origin itself).

This may remind us of a function from the exercises in Spivak, Chapter 2, which is not constant in any neighbourhood of the origin, but on any line through the origin is constant on a interval about the origin.

**2**. Prove Theorem 2 in the following special case: assume that $f$ has a local maximum at a point $x \in M$, and moreover that there exists an open set $U \subset \mathbb{R}^n$ containing $x$ such that $U \cap M = \{x \in U : x^{n-k+1} = \cdots x^n = 0\} = U \cap (\mathbb{R}^{n-k} \times \{0\})$.

**remark**: Although $M$ has a very simple explicit form near $x$, you will still need to use the assumption, imported from Theorem 1, that $g'(x)$ has rank $k$.

**3**. Prove Theorem 2. You may use Problem **2** above if you like.

**4**. (based on Problem 3-6 of Spivak.) Let $A$ be a rectangle in $\mathbb{R}^n$, and let $f$ be a real-valued function on $A$.

(a) If $f$ is integrable, show that also $|f|$ is integrable.

(b) Is the converse also true? why/why not? Please give a proof or a counterexample.