Dynamics of topological defects in nonlinear field theories

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Abstract.

We survey recent results which establish that characterize the dynamics, in a certain asymptotic limit, of interfaces in certain semilinear hyperbolic equations, as well as vortex filaments in semilinear hyperbolic systems. This survey includes a lengthy discussion of heuristic considerations, together with some complete proofs in simple model cases. We also present some novel recent approaches to problem that geometric evolution problem of timelike extremal submanifolds of Minkowski space, which governs the asymptotic dynamics of interfaces and vortex filaments.

§1. Introduction

In this paper, we survey some recent results that establish links between certain semilinear hyperbolic equations, whose prototype is

$$(1) \qquad u_{tt} - \Delta u + \frac{1}{\epsilon^2}(|u|^2 - 1)u, \qquad u : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^k$$

with $N > k \in \{1, 2\}$ and $0 < \epsilon \ll 1$, and hyperbolic geometric evolution equations, exemplified by the problem

$$(2) \qquad \text{Minkowskian mean curvature} = 0$$

for (timelike) submanifolds of codimension k in Minkowski space $\mathbb{R}^{1+N}$. We will also survey some recent results about (2), which is of great interest in its own right. For a curve that sweeps out a (1+1)-dimensional worldsheet, (2) is exactly the dynamical law associated to the celebrated Nambu-Goto action [51, 26].

The links between the PDEs and the corresponding geometric problems can be stated, imprecisely, in several distinct but closely related ways. For example,

- most level sets of suitable solutions of a PDE are close to a solution of the associated geometric problem.

- for suitable solutions of a PDE, energy concentrates around a submanifold solving the associated geometric problem.

- suitable solutions of a PDE exhibit an interface (in the scalar case $k = 1$) or a "vortex submanifold" (for $k = 2$) near a submanifold solving a geometric problem.

The motivations for these questions come both from the large literature on parallel questions about elliptic and parabolic equations, and from certain problems arising in mathematical physics. These are discussed in the latter part of this introduction.

Following that, Sections 2 - 4 discuss the asymptotic behaviour of interfaces in the *scalar* case of (1) and somewhat more general equations. The goal of this discussion is to make recent results of [35, 24], summarized in Section 3, accessible to people who do not have much prior knowledge of hyperbolic equations or (semi-) Riemannian geometry. Thus, Section 2 is devoted to heuristic arguments, and Section 4 gives a complete and detailed proof of a simple model theorem[1], together with a discussion of modifications needed to this basic argument in order to establish more satisfactory results. Many elements of the formal arguments of Section 2 reappear in the proofs of Section 4.

In Section 5, we state results from [35, 18] describing dynamics of codimension 2 defects (sometimes called strings or vortex filaments) in $\mathbb{R}^2$-valued solutions of (1), as well as in a more complicated equation of the same general character called the Abelian Higgs model. The proofs of these results are discussed very briefly in Section 6.

Section 7 discusses various aspects of the Minkowski extremal surface problem (2). This is intended both to provide background to the results discussed in Sections 2 - 6, and also to survey recent some papers, including [6, 8, 36, 53], which suggest a number of interesting directions for future research. Finally, in Section 8 we describe some open problems.

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[1]See Theorem 2. Although simple, this result is new, since it addresses a larger family of equations than is considered in [35, 24], see (8).

The author remains grateful for the invitation to a very stimulating and well-organized meeting.

1.1. background: elliptic and parabolic

Elliptic versions of problems similar to the ones that we will study were first examined in the context of the calculus of variations, and characterize the $\epsilon \to 0$ limit, in the sense of Γ-convergence, of the sequence of functionals

$$u \in H^1(\Omega; \mathbb{R}) \mapsto E_\epsilon(u) := \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{2\epsilon}(|u|^2 - 1)^2$$

for $\Omega \subset \mathbb{R}^N, N \geq 2$. These results (with the basic theory essentially established in [50], and further developed in papers such as [49, 63]) imply in particular that, roughly speaking, if $u_\epsilon \in H^1(\Omega)$ satifies

$$(3) \qquad\qquad -\Delta u_\epsilon + \frac{1}{\epsilon^2}(|u_\epsilon|^2 - 1)u_\epsilon = 0,$$

and if in addition u_ϵ is a minimizer of E_ϵ, subject to suitable boundary conditions, then (after possibly passing to a subsequence) u_ϵ converges in L^1 to a limiting function $u \in BV(\Omega)$ such that $|u| = 1$ a.e. and $\Gamma := \partial\{x \in \Omega : u(x) = 1\}$ is an area-minimizing hypersurface in Ω, at least in a weak sense. In particular, Γ (weakly) satisfies

$$(4) \qquad\qquad \text{mean curvature} = 0.$$

Refined results related to the Γ-convergence of E_ϵ continue to be a topic of current interest, with important contributions in [30, 57]. A different family of arguments (see for example [55, 20]) employ Liapunov-Schmidt reduction and related arguments, relying ultimately on the implicit function theorem and control of the spectrum of some linearized operator, to build solutions of (3) that are close to a given nondegenerate minimal surface (4). These arguments yield existence results that give very precise descriptions of the solutions that are constructed.

In a different direction, results in [40, 1] consider vector-valued functions $u \in H^1(\Omega; \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^N, N \geq 3$, and establish Γ-convergence results characterizing the asymptotic behaviour of the functionals E_ϵ as $\epsilon \to 0$. These results imply, for example, that for suitable energy-minimizing sequences of $\mathbb{R}^2$-valued solutions of (3), the energy concentrates, as $\epsilon \to 0$, around a codimension 2 surface Γ satisfying (4), at least in a weak sense. A different approach to this question relies on PDE techniques and therefore, for the strongest results (see for example [10]), applies to arbitrary sequences of solutions (not necessarily energy-minimizing) with appropriate uniform energy bounds.

There is a similarly long history of results that establish relationships between semilinear parabolic equations such as

$$(5) \qquad u_t - \Delta u + \frac{1}{\epsilon^2}(|u|^2 - 1)u = 0, \qquad u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^k$$

with $N > k \in \{1, 2\}$ and $0 < \epsilon \ll 1$, and geometric flows such as

$$(6) \qquad\qquad \text{velocity} = \text{ mean curvature}$$

for codimension k submaifolds of $\mathbb{R}^N$.

In the scalar case, these results show, roughly speaking, that for suitable initial data, solutions of (5) exhibit an interface whose evolution is governed by (6). This was first proved using linearization techniques [19], which yield quite a detailed description of solutions but are valid only locally in time. A number of different proofs followed, including maximum principle arguments (see [16, 21, 5]), and measure theoretic methods combined with parabolic estimates as in Ilmanen [31]. These later arguments give less precise descriptions of the solutions of (5) than the earlier work of [19], but they make possible results that are valid globally in t, with (6) understood in a suitable weak sense.

A little later, in a 1995 lecture series [62], Soner presented a new perspective on the relationship between (5) and (6). His argument relies on a rather straightforward but remarkable computation of

$$(7) \qquad\qquad \frac{d}{dt} \int_{\mathbb{R}^N} \zeta \; e_\epsilon(u)dx$$

where $e_\epsilon(u)$ is a natural energy density associated with a solution u of (5), and ζ is a smooth function such that $\zeta(t, x) = \frac{1}{2}\operatorname{dist}(x, \Gamma_t)^2$ near Γ_t, where the latter solves (6). This argument did not improve much on earlier results about the scalar case — indeed, the results of [19, 31, 21, 5] are in some ways considerably stronger — but led to the the first results [39, 43] relating (5) and (6) in the $k = 2$ case; these were essentially adaptations to the vector setting of Soner's weighted energy estimates [62]. These results, valid only locally in time, show that for solutions of (5) with suitable energy bounds, energy concentrates around a codimension 2 submanifold Γ satisfying (6). Proving that this behavoiur holds globally in t turned out to be a very difficult question, finally completely settled in [11] following partial results of [4] and others. These authors employ a combination of PDE and measure theoretic arguments, along lines pioneered by Ilmanen [31] in the scalar case, and obtain results valid globally in time, phrased in the language of varifold convergence, describing the precise way in which energy concentrates

around the codimension 2 surface Γ. A very nice review of some of the most significant results relating (5) and (6) is given in [12], with explicit attention to similarities and differences between the analysis in the scalar and (more subtle) vector cases.

Finally, we remark that an important conjecture holds that for $0 < \epsilon \ll 1$, vortex filaments in solutions of the Gross-Pitaevskii equation

$$iu_t - \Delta u + \frac{1}{\epsilon^2}(|u|^2 - 1)u = 0, \qquad u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{C}$$

with suitable initial data approximately evolve, at least locally in time, by a Hamiltonian evolution equation called the binormal curvature flow:

$$\text{velocity} = \textit{binormal} \text{ curvature}.$$

This is the Schrödinger variant of the family of problems that we consider here. Some partial results in this direction are proved in [38], but overall, this problem is very much open, and we will not discuss it in this paper.

The literature on the $\epsilon \to 0$ limit of (3) and (5) generally relies very heavily on tools such as maximum principles (for scalar equations) or powerful elliptic and parabolic estimates, and thus does not suggest any plausible strategies for studying similar questions about hyperbolic equations such as (5). There are only two partial exceptions to this rule. First, it is possible that linearization arguments might be made work in the hyperbolic setting (see Problem 6 in Section 8 below) although the requisite estimate appear to be hard to obtain. Second, although the weighted energy estimates estimates of Soner [62] certainly use the parabolic character of (5), in retrospect one can see that they do not do so in an essential way. Indeed, it turns out our main estimates can be viewed as hyperbolic variants of these weighted energy estimates.

1.2. some physical models

The equations considered here arise in various branches of physics, including the study of superconductivity, superfluidity, and phase boundaries in materials, sometimes as toy versions of more complicated physical models. The hyperbolic problems that we focus on date back to 1970, when Nambu proposed that string dynamics might be governed by an action functional which is exactly to the Minikowsian area swept out by the string as it evolves, see (110). As related by Goddard [25], "Nambu's initial discussion of this action was in notes [51] prepared for a symposium in Copenhagen in August 1970 that, in the event, he was unable to attend. (Although knowledge of this aspect of the content of the notes

spread by word of mouth over time, the notes were not generally available until the publication of Nambu's Selected Papers in 1995.)" A few months after the Copenhagen conference, work of Goto [26] elaborated on some properties of Nambu's proposed action functional, and this lead to it eventually becoming known as the Nambu-Goto action. Some aspects of this action are discussed in Section 7. The discussion there relies heavily on a now-standard choice of gauge that was another important contribution of Nambu, in joint work with Mansouri [46] which clarified slightly earlier work of Chang and Mansouri [15].

Shortly after Nambu's proposal, Nielsen and Olesen, who were interested in obtaining the Nambu-Goto action from a field theory, argued [54] that certain semilinear wave equations arising from gauge theory should have solutions that exhibit vortex lines that, in their words, "can approximately be identified with the Nambu string." They considered in detail the example of the Abelian Higgs model, see (82), (83), which is essentially a complex-valued wave equation (1) coupled to electromagnetic fields. Some aspects of the Nielsen-Olesen picture have been rigorously established in [18], as we describe in Section 5.2, and there remain a number of associated open problems; see Section 8.1. It is worth noting that the Nielsen and Olesen were explicitly motivated by the phenomenon of vortex lies in superconductors, which was by then already well-understood.

The study of dynamics of defects in semilinear wave equations received a major impetus from the work of Kibble [42], who noted that the ideas explored in [54], as well as in earlier work on elliptic and parabolic problems describing superconductivity and ferromagnetic materials, may be relevant to descriptions of the large-scale structure of spacetime, and that these ideas suggest the possible existence of structures he dubbed cosmic strings or domain walls. A nonlinear wave equation of roughly the form (1), but with a somewhat different nonlinearity, has also been suggested in the cosmological literature as a model for what is called the decay of a false vacuum, see Coleman [17].

As far as we know, the first work in the applied mathematics literature on defects in hyperbolic equations was a paper of Neu [52] that gave a formal analysis of interfaces for a scalar semilinear wave equation of the form (1). Neu's formal analysis has been elaborated on and extended to more complicated equations in [59, 58].

Some rigorous but conditional results about the $\epsilon \to 0$ limit of equation (1) are given in [7]. Dynamics of point defects in various semilinear hyperbolic equations (such (1) when $N = k = 2$ or the Abelian Higgs model when $N = 2$) has been analyzed in works such as [28, 34, 44, 64, 65].

§2. heuristic arguments

In this section we give a formal analysis of the equation[2]

$$(8) \qquad c^2 u_{tt} + d^2 u_t - \Delta u + \frac{1}{\epsilon^2} f(u) = 0, \qquad u : (0, T) \times \mathbb{R}^n \to \mathbb{R},$$

where at least one of c, d is nonzero and f has the form $f = F'$, for a smooth $F : \mathbb{R} \to \mathbb{R}$ such that

$$(9) \qquad F(\pm 1) = 0, \qquad F(x) > 0 \text{ if } |x| \neq 1, \qquad F''(\pm 1) > 0.$$

This discussion thus includes as special cases both the pure parabolic (5) and hyperbolic (1) problems. The standard cubic nonlinearity from the introduction corresponds to the choice $F(s) = \frac{1}{4}(1 - s^2)^2$.

The formal considerations we discuss here also apply the elliptic case $c = d = 0$, provided only that the domain of u is taken to be an open subset of $\mathbb{R}^n$.

2.1. a formal analysis

We first seek to build a family of approximate solutions $(U_\epsilon)_{0 < \epsilon \leq 1}$ of (8), of the form

$$(10) \qquad U_\epsilon(t, x) = q\left(\frac{\delta(t, x)}{\epsilon}\right)$$

for $q : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R}^{1+n} \to \mathbb{R}$, both independent of ϵ, to be determined. We do this in the hope, to be justified later, that one can find exact solutions that are close in suitable ways to the functions we construct here.

To proceed, we substitute (10) into the left-hand side of (8), expand, and collect terms of the same order to obtain

$$c^2 U_{\epsilon, tt} + d^2 U_{\epsilon, t} - \Delta U_\epsilon + \frac{1}{\epsilon^2}(U_\epsilon^2 - 1)U_\epsilon$$

$$= \frac{1}{\epsilon^2}\left[q''\left(\frac{\delta}{\epsilon}\right)(c^2(\partial_t\delta)^2 - |\nabla\delta|^2) + f\left(q\left(\frac{\delta}{\epsilon}\right)\right)\right]$$

$$+ \frac{1}{\epsilon}q'\left(\frac{\delta}{\epsilon}\right)(c^2\partial_{tt} - \Delta + d^2\partial_t)\delta.$$

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[2]We have written (8) in a way that makes it easy to treat the pure parabolic case as a $c \to 0$ limit of damped hyperbolic equations. It would however be more standard to write it in the from $u_{tt} + \tilde{d}^2 u_t - \tilde{c}^2 \Delta u + \frac{1}{\epsilon^2}f(u) = 0$. In particular, as a result of our violation of standard conventions, loosely speaking, the "speed of light" in (8), and hence throughout our discussion, is c^{-1} rather than c.

The terms of order $\frac{1}{\epsilon^2}$ vanish if

$$(11) \qquad\qquad\qquad -q'' + f(q) = 0$$

and

$$(12) \qquad\qquad\qquad -c^2(\partial_t \delta)^2 + |\nabla \delta|^2 = 1.$$

If in addition

$$(13) \qquad (c^2 \partial_{tt} - \Delta + d^2 \partial_t)\delta \approx 0 \qquad \text{wherever } q'(\tfrac{\delta}{\epsilon}) \text{ is large,}$$

then the terms of order $\frac{1}{\epsilon}$ may be considered negligible, in which case we can consider U_ϵ to be a good approximate solution of (8).

   We conclude that if q solves (11), and if d solves (12), (13), then U_ϵ as defined in (10) looks like a reasonable approximate solution.

   We will see that, roughly speaking, (11) and (12) determine the shape of the interface, and (13) determines the dynamics of the interface.

   We will first show that such approximate solutions exist and gain a detailed understanding of what they look like, by considering (11), (12) and (13) in turn. A later goal will be to prove that certain actual solutions of (8) are in fact close to these approximate solutions.

   We will consider (11) together with the additional conditions

$$(14) \qquad\qquad q(0) = 0, \qquad q(s) \to \pm 1 \quad \text{as } s \to \pm\infty.$$

For such q, the ansatz (10) then implies that U_ϵ has an interface near $\Gamma := \{(t,x) : \delta(t,x) = 0\}$, by which we mean that

$$U_\epsilon \approx 1 \text{ where } \delta(t,x) \gg \epsilon, \qquad U_\epsilon \approx -1 \text{ where } \delta(t,x) \ll -\epsilon.$$

The set Γ is thus expected to play a central role in our analysis. We will often use the notation

$$(15) \qquad\qquad\qquad \Gamma_t := \{x \in \mathbb{R}^n : (t,x) \in \Gamma\}.$$

   To solve (12) and (approximately) (13), we will argue as follows: First, we consider a hypersurface $\Gamma \subset \mathbb{R}^{1+n}$, which we take to be smooth, and we find a function, say δ_Γ, that solves (12) near Γ and in addition satisfies

$$(16) \qquad\qquad\qquad \delta_\Gamma = 0 \text{ on } \Gamma.$$

Having found δ_Γ in this way (at least near Γ), we will then ask what geometric information about Γ is encoded in the expression

$$(c^2 \partial_{tt} - \Delta + d^2 \partial_t)\delta_\Gamma,$$

and in particular, what does it mean for this expression to vanish everywhere on Γ. This will enable us to reformulate (13) as a geometric evolution equation for Γ.

The differential operator $c^2\partial_{tt} - \Delta + d^2\partial_t$ has a hybrid character, in that the second-order part $c^2\partial_{tt} - \Delta$ is naturally associated to a metric $(\eta_{\alpha\beta})$ described below, and is invariant with respect to maps that preserve this metric, whereas the first-order part $d^2\partial_t$ does not have these properties. So parts of our discussion will refer heavily to the metric, whereas other parts will neglect it completely.

2.2. the profile q

Multiplying (11) by q', integrating, and using properties (9) of F, we find that (11) and (14) hold if and only if

$$(17) \qquad\qquad q' = \sqrt{2F}, \qquad q(0) = 0,$$

and it is clear that this first-order equation has a unique solution. Since we have assumed that $F'' > 0$ at ± 1, this solution also satisfies

$$(18) \qquad |q(s) - \operatorname{sign}(s)| \le Ce^{-c|s|} \qquad \text{as } s \to \pm\infty.$$

The profile q is characterized by an optimality property. Indeed, for any $\widetilde{q} \in H^1_{loc}(\mathbb{R})$ such that $\widetilde{q}(s) \to \pm 1$ as $s \to \pm\infty$,

$$(19)\ \ c_0 := \int_{-1}^{1} \sqrt{2F}\, ds \ \ = \ \int_{-\infty}^{\infty} \sqrt{2F(\widetilde{q})}\, \widetilde{q}'(s)\, ds \ \le \int_{-\infty}^{\infty} \frac{\widetilde{q}'^2}{2} + F(\widetilde{q})\, ds.$$

Moreover, equality holds if and only if $\widetilde{q}' = \sqrt{2F(\widetilde{q})}$, and this equation characterizes q up to translation.

Since $\frac{1}{2}q'^2 = F(q)$ pointwise, it is easy to check that for any $\epsilon > 0$ and $r > 0$, if we define $q_r(s) := q(\frac{s}{r})$, then

$$(20) \qquad\qquad \int_{-\infty}^{\infty} \frac{\epsilon}{2}q_r'^2 + \frac{1}{\epsilon}F(q_r)\, ds = (\frac{\epsilon}{2r} + \frac{r}{2\epsilon})c_0.$$

Thus for this scaled energy, the parameter ϵ fixes a length-scale: a dilation of the basic profile q by exactly the factor $r = \epsilon$ is energetically optimal.

2.3. solution of the eikonal equation for $c \ne 0$

Now consider (12), (16) with some fixed *nonzero c*.
We will write

$$(21) \quad (\eta_{\alpha\beta}) = \operatorname{diag}(-c^{-2}, 1, \ldots, 1), \qquad (\eta^{\alpha\beta}) = \operatorname{diag}(-c^2, 1, \ldots, 1),$$

where α, β run from 0 to n. Thus $(\eta_{\alpha\beta})$ is a form of the Minkowski metric on $\mathbb{R}^{1+n}$, and we will view $(\eta^{\alpha\beta})$ as the dual inner product on 1-forms. We implicitly sum over repeated upper and lower indices. We will sometimes write $x = (x^0, \ldots, x^n)$ to denote a point in $\mathbb{R}^{n+1}$, and we may write x^0 and t interchangeably. We will also sometimes use the notation $x = (t, \vec{x})$, with $\vec{x} \in \mathbb{R}^n$.

We use upper indices such as v^α or v^β to denote the components of a vector, and lower indices v_α or v_β to denote the components of a covector. The metric $(\eta_{\alpha\beta})$ induces a natural isomorphism between the spaces of vectors and covectors, and given a vector with components v^α, the associated covector is defined by $v_\alpha := \eta_{\alpha\beta}v^\beta$. Similarly, if v_α are components of a covector, then $v^\alpha := \eta^{\alpha\beta}v_\beta$ denote the components of the associated vector.

In particular, given a function f the differential df (a covector) has components $\partial_{x^\alpha}f$, and the gradient $\operatorname{grad}f$ (a vector) has components $\eta^{\alpha\beta}\partial_{x^\beta}f$.

Lemma 1. *Assume that $\Gamma \subset \mathbb{R}^{1+n}$ is a smooth hypersurface and that for every $x \in \Gamma$, there exists a vector $\nu = \nu(x)$ such that $x \in \Gamma \mapsto \nu(x)$ is continuous, and*

$$\tag{22} \eta_{\alpha\beta}\nu^\alpha\tau^\alpha = 0 \quad \text{for all } \tau^\alpha \in T_x\Gamma,$$

and

$$\tag{23} \eta_{\alpha\beta}\nu^\alpha\nu^\beta = 1.$$

Then there is a neighborhood $\mathcal{N}$ of Γ in which there exists a smooth solution δ_Γ of (12), (16). In addition, for every $x \in \Gamma$,

$$\tag{24} \delta_\Gamma(x + s\nu(x)) = s$$

for all s in a neighborhood of 0.

Below, we briefly recall the (standard) proof of Lemma 1, to emphasize that some aspects of our later arguments arise naturally from simple considerations involving the first-order equation (12). First we give some definitions:

We say that Γ is *timelike* at $x \in \Gamma$, with respect to the metric $(\eta_{\alpha\beta})$, if there exists a vector $n(x)$ satisfying (22), (23), and that Γ is *timelike* if it is timelike at every point x. One can verify that Γ is timelike at x if and only if $|v| < \frac{1}{c}$, where $v(x)$ is the velocity of Γ at x (see (42) below for the definition.)

We say that a vector v is *spacelike* with respect to the metric $(\eta_{\alpha\beta})$ if $\eta_{\alpha\beta}v^\alpha v^\beta > 0$, and *timelike* if $\eta_{\alpha\beta}v^\alpha v^\beta < 0$. Geometrically, (22), (23)

state that $\nu(x)$ is a spacelike unit normal to $T_x\Gamma$ with respect to the $(\eta_{\alpha\beta})$ metric. Since the space of normal vectors at any point x is 1-dimensional, if Γ is timelike then at every $x \in \Gamma$ there are exactly two normal vectors satisfying (23), and a continuous choice of $\nu(x)$ is possible exactly when Γ is orientable — this can be taken as the definition of orientable.

We interpret δ_Γ as the (signed) distance to Γ with respect to the metric $(\eta_{\alpha\beta})$. Indeed, $x + s\nu(x)$ is a point reached by starting at $x \in \Gamma$ and moving a (signed) distance s in the direction $\nu(x)$ normal to Γ at x (where "distance" and "normal" are both understood with respect to the metric $(\eta_{\alpha\beta})$). According to (24), at this point the value of δ_Γ is just the distance parameter s.

Proof of Lemma 1. It is helpful to rewrite (12) in the form

$$(25) \quad F(x, \delta_\Gamma, d\delta_\Gamma) = 0, \qquad \text{for } F(x, z, p) = F(p) := \frac{1}{2}\left(\eta^{\alpha\beta}p_\alpha p_\beta - 1\right),$$

where $(\eta^{\alpha\beta}) := \mathrm{diag}(-c^2, 1, \ldots, 1)$. Following the standard method of characteristics (see for example [22], section 3.2), we consider the systems of ODEs

$$(26) \quad \frac{d}{ds}x^\alpha = F_{p_\alpha}(x, z, p) = \eta^{\alpha\beta}p_\beta \qquad\qquad \alpha = 0, \ldots, n$$

$$(27) \quad \frac{d}{ds}z = p_\alpha F_{p_\alpha}(x, z, p) = \eta^{\alpha\beta}p_\alpha p_\beta$$

$$(28) \quad \frac{d}{ds}p_\alpha = -p_\alpha F_z(x, z, p) - F_{x^\alpha}(x, z, p) = 0, \qquad \alpha = 0, \ldots, n.$$

We fix some $x \in \Gamma$, and we solve (26) - (28) with initial data

$$x(0) = x, \qquad z(0) = \delta_\Gamma(x) = 0, \qquad p_\alpha(0) = \nu_\alpha(x) = \eta_{\alpha\beta}\nu^\beta(x),$$

where ν^β is the (spacelike) unit normal to Γ. With this choice, it is clear that $F(x(0), z(0), p(0)) = F(p(0)) = 0$, and also that the data satisfies the standard compatibility condition, which in this case reduces to the requirement that p_α be conormal to the surface Γ on which the solution we seek is constant. Indeed, these conditions are equivalent to (23) and (22) respectively.

It is almost immediate that the solution of (26) - (28) is given by

$$(29) \quad p_\alpha(s) = \nu_\alpha(x), \qquad z(s) = s, \qquad x(s) = x + s\nu(x).$$

Standard facts about characteristics (*i.e.*, a short argument using the inverse function theorem) then guarantees that the map $(x, s) \in \Gamma \times \mathbb{R} \mapsto$

$x + s\nu(x)$ is invertible in a neighborhood of $(x, 0)$ for every $x \in \Gamma$, and that in this neighborhood, the condition

$$\delta_\Gamma(x(s)) = z(s) = s \qquad \text{along each solution (29)},$$

yields a well-defined function δ_Γ that satisfies (12), (16), and (24).

<div align="right">Q.E.D.</div>

Remark 1. *In the argument sketched above, it is natural to parametrize a subset of Γ by an embedding $\Psi : O \to \mathbb{R}^{1+n}$, where O is an open subset of $\mathbb{R}^n$. It is often convenient to assume that Ψ has the form*

$$(30) \qquad \Psi(y^0, \dots, y^{n-1}) = \left(y^0, \psi(y^0, \dots, y^{n-1})\right),$$

and that

$$(31) \qquad \partial_{y^0}\psi \cdot \partial_{y^i}\psi = 0, \qquad i \in \{1, \dots, n-1\}.$$

Then the y^0 coordinate on O corresponds to time, and $\psi(t, \cdot)$ parametrizes (a subset of) Γ_t. Having fixed Ψ, we may define

$$(32) \qquad \phi(y^0, \dots, y^n) = \Psi(y^0, \dots, y^{n-1}) + y^n \, \nu\left(\Psi(y^0, \dots, y^{n-1})\right).$$

Then ϕ is a diffeomorphism of a neighborhood in $\mathbb{R}^n \times \mathbb{R}$ of $O \times \{0\}$ onto its image, which is a neighborhood of Γ — this is the local invertibility claim mentioned in the proof above. Equivalently, we can view $(y^0, \dots, y^n)$ as defining a local coordinate system, and we will sometimes refer to these as normal coordinates *near Γ.*

It follows from (24) and (32) that

$$(33) \qquad \delta_\Gamma \circ \phi(y^0, \dots, y^n) = y^n,$$

and hence that $U_\epsilon = q(\delta_\Gamma / \epsilon)$ satisfies

$$(34) \qquad U_\epsilon \circ \phi(y^0, \dots, y^n) = q\left(\frac{y^n}{\epsilon}\right).$$

Thus U_ϵ has a particularly simple form in the $(y^0, \dots, y^n)$ coordinates. Note also that (33) states exactly that

$$(35) \qquad \delta_\Gamma = (\phi^{-1})^n \qquad \text{(i.e., the nth component of } \phi^{-1}).$$

2.4. solution of the eikonal equation for $c = 0$

When $c = 0$, the problem (12), (16) reduces to

$$(36) \qquad \delta^{ij}\, \partial_{x^i} \delta_\Gamma\, \partial_{x^j} \delta_\Gamma = 1 \quad \text{near } \Gamma \subset \mathbb{R}^{1+n}, \qquad \delta_\Gamma = 0 \text{ on } \Gamma$$

where i, j are summed implicity $1, \dots, n$. Parallel to Lemma 1 we have

Lemma 2. *Assume that $\Gamma \subset \mathbb{R}^{1+n}$ is a smooth hypersurface and that for every $x \in \Gamma$, there exists a vector $\nu = \nu(x)$ such that $x \in \Gamma \mapsto n(x)$ is continuous, and*

$$(37) \qquad \nu^0 = 0, \qquad \delta_{ij}\nu^i\tau^j = 0 \;\; \text{for all } \tau^\alpha \in T_x\Gamma, \qquad \delta_{ij}\nu^i\nu^j = 1.$$

Then there is a neighborhood $\mathcal{N}$ of Γ in which there exists a smooth solution δ_Γ of (36), and for every $x \in \Gamma$,

$$(38) \qquad \delta_\Gamma(x + s\nu(x)) = s \qquad \text{for all } s \text{ near } 0$$

Now (37) states that at $x = (t, \vec{x}) \in \Gamma$, the vector $\nu(x)$ is the Euclidean unit normal in $\mathbb{R}^n$ to Γ_t at $\vec{x}$.

We will sometimes abuse terminology and say that, in the case $c = 0$, Γ is timelike if it has finite velocity everywhere.

The proof of the lemma, which we omit, can be established by the same procedure as in the case $c > 0$, or (at least formally) by taking the $c \searrow 0$ limit of Lemma 1.

2.5. the geometry of Γ

We now consider (13). More precisely, we seek timelike hypersurfaces Γ such that

$$(39) \qquad (c^2\partial_{tt} - \Delta + d^2\partial_t)\delta_\Gamma = 0 \qquad \text{on } \Gamma$$

where δ_Γ is the solution of the eikonal equation (12), (16) that we have found above. We will proceed by making the geometric content of (39) more transparent, and then arguing that, at least in some interesting special cases, it reduces to geometric evolution equations for which we can obtain solutions by appealing to known well-posedness results.

In this discussion, we will treat quantities such as mean curvature, velocity, and acceleration as (signed) scalars whose sign depends on a choice of the unit normal ν. For us this choice is encoded in the choice of sign of δ_Γ. Specifically, our convention is that the mean curvature of a hypersurface Γ can be computed (in any Lorentzian or Riemannian manifold) by

$$(40) \qquad H_\Gamma(x) := -\operatorname{div} \bar{\nu}(x), \qquad x \in \Gamma$$

where $\bar{\nu}$ is any smooth unit vector field such that $\bar{\nu} = \nu = \operatorname{grad} \delta_\Gamma$ on Γ. Here "unit", "normal", "gradient", and so on are understood with respect to the relevant metric. For the convenience of the reader, we recall the definition of mean curvature and give the proof of (40) in Section 7.5.

We will write H_Γ^m to denote the mean curvature of Γ with respect to the Minkowski metric $(\eta_{\alpha\beta})$, and $H_{\Gamma_t}^e$ to denote the Euclidean mean curvature of a time-slice Γ_t as a submanifold of $\mathbb{R}^n$.

To define velocity v and acceleration a, let I be a open interval and $\gamma : I \to \mathbb{R}^n$ a map such that

$$(41) \qquad \gamma(t) \in \Gamma_t \qquad \dot{\gamma}(t) \perp T_{\gamma(t)}\Gamma_t \quad \text{for all } t \in I,$$

where here $\perp$ is understood with respect to the Euclidean metric. Then for any $t \in I$, writing $x := (t, \gamma(t)) \in \Gamma$ we define

$$(42) \qquad v(x) = \dot{\gamma}(t) \cdot \frac{\nabla \delta_\Gamma}{|\nabla \delta_\Gamma|}(x), \qquad a(x) = \ddot{\gamma}(t) \cdot \frac{\nabla \delta_\Gamma}{|\nabla \delta_\Gamma|}(x).$$

where $\nabla \delta_\Gamma$ denotes the spatial gradient $(\partial_{x^1}\delta_\Gamma, \ldots, \partial_{x^n}\delta_\Gamma)$. We will prove

Lemma 3. *Assume that δ_Γ solves the eikonal equation*

$$-c^2(\partial_t \delta_\Gamma)^2 + |\nabla \delta_\Gamma|^2 = 1$$

for some $c \geq 0$, and let $\Gamma := \{x : \delta_\Gamma = 0\}$.
   Then for $x = (t, \vec{x}) \in \Gamma$,

$$(43) \qquad (c^2 \partial_{tt} - \Delta)\delta_\Gamma(x) = \begin{cases} H_\Gamma^m(x) & \text{if } c > 0 \\ H_{\Gamma_t}^e(\vec{x}) & \text{if } c = 0, \end{cases}$$

and

$$(44) \qquad \partial_t \delta_\Gamma(x) = \frac{-v}{(1 - c^2 v^2)^{1/2}}.$$

   Moreover, if $c > 0$ then

$$(45) \qquad H_\Gamma^m = (1 - c^2 v^2)^{-1/2}\left(H_{\Gamma_t}^e - \frac{c^2}{1 - c^2 v^2}a\right).$$

A vectorial analog of (45) for timelike $1 + 1$-dimensional surfaces of arbitrary codimension is proved in [6]. In fact we will barely use (45) in the sequel, but it provides an interesting interpretation of H_Γ^m.

Before giving the proof, we remark that the Lemma implies that if $d = 0$ then (39) is exactly the Minkowski extremal surface problem, that is, the condition that the Minkowski mean curvature vanishes; and if $c = 0$ then it exactly the mean curvature flow. In both cases, short-time existence of smooth solutions for smooth data is well-known; see for example [45, 48, 13] in the Minkowski case.

In the general case, (39) is a damped extremal surface problem. We do not know of any well-posedness results for this equation, but we suppose that such results can be proved by rather standard techniques.

Proof. Formula (43) is well-known when $c = 0$, so we fix $c > 0$, and we simply recall the standard reasoning from the $c = 0$ case, which goes as follows: Let $\bar{\nu}$ be the vector field defined by $\bar{\nu} := \operatorname{grad} \delta_\Gamma$, so that $\bar{\nu}^\alpha = \eta^{\alpha\beta} \partial_{x^\beta} \delta_\Gamma$. It follows directly from the eikonal equation that $\eta_{\alpha\beta} \bar{\nu}^\alpha \bar{\nu}^\beta = 1$ wherever it is defined, and it is generally true, and easy to check, that $\operatorname{grad} \delta_\Gamma$ is orthogonal to any level set of δ_Γ, including in particular the level set $\Gamma := \{\delta_\Gamma = 0\}$. Then using (40)

$$H_\Gamma^m(x) = -\operatorname{div} \bar{\nu}(x) = -\partial_{x^\alpha} \bar{\nu}^\alpha = -\eta^{\alpha\beta} \partial_{x^\alpha \, x^\beta} \delta_\Gamma,$$

proving (43) for $c > 0$.

Next, we fix a curve γ as in (41), and we differentiate the identity $\delta_\Gamma(t, \gamma(t)) = 0$ to find that

$$(46) \qquad \partial_t \delta_\Gamma(t, \gamma(t)) + \nabla \delta_\Gamma(t, \gamma(t)) \cdot \dot{\gamma} = 0.$$

Recalling the definition (42) of v and the fact that $\dot{\gamma}(t)$ is normal to Γ_t, we deduce that

$$v = -\frac{\partial_t \delta_\Gamma}{|\nabla \delta_\Gamma|}, \qquad \text{and} \quad \dot{\gamma} = -\frac{\partial_t \delta_\Gamma}{|\nabla \delta_\Gamma|^2} \nabla \delta_\Gamma \; = \; -v \frac{\nabla \delta_\Gamma}{|\nabla \delta_\Gamma|}.$$

Then we can rewrite the eikonal equation to find that

$$(47) \qquad |\nabla \delta_\Gamma|^2 = (1 - c^2 v^2)^{-1}.$$

Combining these identities yields (44).

The rest of the proof is a computation leading to (45) and can be skipped without much loss. First, by differentiating the eikonal equation we obtain

$$(48) \quad c^2 \partial_{tx^i} \delta_\Gamma \; \partial_t \delta_\Gamma = \partial_{x^i x^j} \delta_\Gamma \; \partial_{x_j} \delta_\Gamma, \qquad c^2 \partial_{tt} \delta_\Gamma \; \partial_t \delta_\Gamma = \partial_{tx^j} \delta_\Gamma \; \partial_{x_j} \delta_\Gamma.$$

By (40) (with respect to the Euclidean metric on $\{t\} \times \mathbb{R}^n$),

$$H^e_{\Gamma_t}(x) = -\partial_{x^i} \left(\frac{\partial_{x^i} \delta_\Gamma}{|\nabla \delta_\Gamma|} \right)$$

$$= -\frac{1}{|\nabla \delta_\Gamma|} \left(\Delta \delta_\Gamma - \frac{\partial_{x^i} \delta_\Gamma \; \partial_{x^j} \delta_\Gamma \; \partial_{x^i x^j} \delta_\Gamma}{|\nabla \delta_\Gamma|^2} \right).$$

By using both identities in (48), we can rewrite this as

$$H^e_{\Gamma_t}(x) = \frac{-1}{|\nabla \delta_\Gamma|} \left(\Delta \delta_\Gamma - \frac{c^4 \partial_t \delta_\Gamma \; \partial_t \delta_\Gamma \; \partial_{tt} \delta_\Gamma}{|\nabla \delta_\Gamma|^2} \right) = \frac{-1}{|\nabla \delta_\Gamma|} \left(\Delta \delta_\Gamma - c^4 v^2 \, \partial_{tt} \delta_\Gamma \right).$$

So

(49) $$(c^2 \partial_{tt} - \Delta) \delta_\Gamma = |\nabla \delta_\Gamma| \left(H^e_{\Gamma_t} + c^2 (1 - c^2 v^2) \, \partial_{tt} \delta_\Gamma \right).$$

Next, differentiating (46) gives

$$\partial_{tt} \delta_\Gamma + 2 \partial_t \nabla \delta_\Gamma \cdot \dot{\gamma} + \nabla^2 \delta_\Gamma : \dot{\gamma} \otimes \dot{\gamma} + \nabla \delta_\Gamma \cdot \ddot{\gamma} = 0$$

at $(t, \gamma(t))$ for any curve γ. Expressing $\dot{\gamma}$ in terms of $\nabla \delta_\Gamma$ then again using (48) to convert space derivatives to time derivatives, and recalling the definitions and a and v, this identity becomes

$$a = -\partial_{tt} \delta_\Gamma \; (1 - 2c^2 v^2 + c^4 v^4) = -(1 - c^2 v^2)^2 \partial_{tt} \delta_\Gamma.$$

Substituting this into (49) and using (47), we arrive at (45).   Q.E.D.

2.6. energetic considerations

We record a couple of properties of the approximate solution U_ϵ described above.

First, it is clear that $U_\epsilon(t, \cdot)$ has an interface near Γ_t, as desired.

Second, for $x = (t, \vec{x}) \in \Gamma$ and euclidean unit normal $\nu^e(x) := \nabla \delta_\Gamma / |\nabla \delta_\Gamma|(x)$, since $|\nabla \delta_\Gamma| = (1 - c^2 v^2)^{-1/2}$,

$$\delta_\Gamma(t, \vec{x} + s\nu^e(x)) = s(1 - c^2 v^2)^{-1/2} + O(s^2)$$

so that

$$U_\epsilon(t, \vec{x} + s\nu^e(x)) = q \left(\frac{s}{\epsilon(1 - c^2 v^2)^{1/2}} \right) + O\left(\frac{s^2}{\epsilon^2} \right).$$

A comparison with (20) suggests that if $u_\epsilon \approx U_\epsilon$ and $U_\epsilon(t, \cdot)$ has an interface moving with nonzero velocity, then $u_\epsilon(t, \cdot)$ will not have an energetically optimal structure across the interface, due to the dilation $(1 - c^2 v^2)^{-1/2}$. Hence it may be difficult to control u_ϵ by naive energy methods if $c \neq 0$. This is one way in which the $c = 0$ parabolic case is

easier. indeed, the parabolic weighted energy estimates of Soner [62] in some sense exploit the energetically optimal structure of the profile in the parabolic case.

Note, however, that this difficulty (which is purely heuristic at this point) vanishes if we consider instead the coordinate system discussed in (32), (34). Indeed, in this coordinate system, our formal arguments suggest that the solution should always be close to the ground state of some variational problem, and we can hope to take advantage of this in our analysis.

2.7. a different heuristic argument

For a family of equations depending on a parameter ϵ, such as (8) in the case $c = 1, d = 0$, that are the Euler-Lagrange equations of some family of functionals $\mathcal{L}_\epsilon$, one can argue formally by a "reduced Lagrangian" approach. This is widely used in the physics literature, and has the advantages of flexibility and (often) simplicity.

Consider generally a family of functionals $\mathcal{L}_\epsilon : X \to \mathbb{R}$, where X is some function space. Suppose we want to describe solutions of the associated Euler-Lagrange equations in terms of some possibly simpler objects belonging to a different space Y. (In our example, possibly $X = H^1_{loc}(W)$ for some open subset $W \subset \mathbb{R}^{1+n}$, and Y may be the space of timelike hypesurfaces in W.) We may argue formally as follows:

(1) First construct a suitable family of maps $U_\epsilon : Y \to X$.
(2) Then define a functional $\mathcal{L}'_\epsilon : Y \to \mathbb{R}$ by $\mathcal{L}'_\epsilon = \mathcal{L}_\epsilon \circ U_\epsilon$, and if possible write $\mathcal{L}'_\epsilon = \mathcal{L}'_0 + o(1)$ as $\epsilon \to 0$, for some $\mathcal{L}'_0 : Y \to \mathbb{R}$.
(3) Conclude (?!) that critical points of $\mathcal{L}_\epsilon$ have approximately the form $U_\epsilon \circ \Gamma$, where $\Gamma \in Y$ is a critical point of $\mathcal{L}'_0$.

This argument, dubious though it might seem, often yields surprisingly useful heuristic information if carried out well. To understand why, and what it means to "carry it out well", note that if $\mathcal{L}'_0 \approx \mathcal{L}_\epsilon \circ U_\epsilon$ in a strong enough sense, and if V is a tangent vector to Y, then

$$d\mathcal{L}'_0(\Gamma)(V) \approx d\mathcal{L}_\epsilon(U_\epsilon(\Gamma))(dU_\epsilon(\Gamma)(V)).$$

Thus if $U_\epsilon(\Gamma)$ is constructed as above (so that $d\mathcal{L}'_0(\Gamma) = 0$), then

$$d\mathcal{L}_\epsilon(U_\epsilon(\Gamma))(dU_\epsilon(\Gamma)(V)) \approx 0.$$

Hence $d\mathcal{L}_\epsilon(U_\epsilon(\Gamma)) \approx 0$ (i.e., $U_\epsilon(\Gamma)$ is an approximate solution) **if**

(50) $d\mathcal{L}_\epsilon(U_\epsilon(\Gamma))$ "(approximately) annihilates $\left(\text{Image } dU_\epsilon(\Gamma)\right)^{\perp}$"

assuming for simplicity a Hilbert space structure. Whether or not (50) holds depends on the construction of the maps $U_\epsilon : Y \to X$. Competent

practitioners of this style of argument are able to produce constructions $U_\epsilon : Y \to X$ that satisfy (50), based on intuition or other factors that are normally left implicit. If this is done, then this procedure yields good approximate solutions of the equation $d\mathcal{L}_\epsilon = 0$ (that is, a function U_ϵ such that $d\mathcal{L}_\epsilon$ is small at U_ϵ), and a mathematician might hope to prove that some actual solutions are close to these approximate solutions.

We illustrate this argument for the family of functionals

$$\mathcal{L}_\epsilon(u) := \int \frac{1}{2}(-|u_t|^2 + |\nabla u|^2) + \frac{1}{\epsilon^2}F(u) \ dx \ dt.$$

corresponding to (8) with $c = 1, d = 0$. We would like to describe solutions in terms of evolving interfaces. As suggested above, we may then let Y be the space of timelike hypersurfaces in $W \subset \mathbb{R}^{1+n}$. Given $\Gamma \in Y$, we choose to define

$$U_\epsilon(\Gamma) := q(\frac{\delta_\Gamma}{\epsilon})$$

where q solves (11) and δ_Γ solves the eikonal equation (12), (16).

Of course our earlier formal arguments already suggest that this ansatz is reasonable, and presumably it could also be justified by careful consideration of (50).

We would now like to write $\mathcal{L}'_\epsilon(\Gamma) = \mathcal{L}_\epsilon(U_\epsilon(\Gamma))$ as $\mathcal{L}'_0(\Gamma) + o(1)$. To do this it is convenient to represent a hypersurface Γ as the image of a map $\Psi : O \subset \mathbb{R}^n \to \mathbb{R}^{1+n}$ as described in Remark 1, and to change variables via the diffeomorphism ϕ defined in (32). We assume for convenience that the domain of ϕ contains $O \times (-r, r)$ for some $r > 0$, and that the energy outside $\phi(O \times (-r, r))$ is negligible.

It is useful to introduce the notation

(51) $g_{\alpha\beta} = \eta_{\mu\nu} \, \partial_{y^\alpha}\phi^\mu \, \partial_{y^\beta}\phi^\nu, \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}, \quad g = \det(g_{\alpha\beta})$

where α, β, μ, ν run from 0 to n; and similarly

(52) $\gamma_{ab} = \eta_{\mu\nu} \, \partial_{y^a}\Psi^\mu \, \partial_{y^a}\Psi^\nu, \quad (\gamma^{ab}) = (\gamma_{ab})^{-1}, \quad \gamma = \det(\gamma_{ab})$

where a, b from from $0, \dots, n - 1$.

Writing $V_\epsilon = U_\epsilon(\Gamma) \circ \phi$ and changing variables yields

$$\mathcal{L}'_\epsilon(\Gamma) \ \approx \ \int_{O \times (-r,r)} \left(\frac{\epsilon}{2} g^{\alpha\beta} \partial_{y^\alpha} V_\epsilon \, \partial_{y^\beta} V_\epsilon + \frac{1}{\epsilon} F(V_\epsilon) \right) \ \sqrt{|g|} \ dy^0 \dots dy^n.$$

Now we recall from (34) that $V_\epsilon(y) = q(\frac{y^n}{\epsilon})$. Also, we will see later that $g^{nn} = 1$, $g^{na} = g^{an} = 0$ for $a < n$, and it follows that

$$g(y^0, \dots, y^n) \ = \ \gamma(y^0, \dots, y^{n-1}) + O(|y^n|).$$

Substituting into the right-hand side above and integrating over y^n, we conclude from (19) that

$$\mathcal{L}'_\epsilon(\Gamma) \approx c_0 \int_O \sqrt{|\gamma|} dy^0 \ldots dy^{n-1} =: \mathcal{L}'_0(\Gamma)$$

As we discuss in Section 7.2, $\mathcal{L}'_0(\Gamma)$ is exactly the Minkowskian area of Γ, and its Euler-Lagrange equation is exactly the condition that the Minkowskian mean curvature of Γ vanishes. (In particular, it is not hard to see that $\mathcal{L}'_0$ depends only on Γ, and not on the parametrization Ψ.) Thus we conclude again that interfaces are expected to evolve by the Minkowski extremal surface equation.

2.8. filament dynamics and systems of equations

The above heuristic considerations extend, albeit somewhat less persuasively, to *systems* of equations such as the Abelian Higgs model, see (82), (83) below. We will give no details about this, but make only a few remarks.

First, these arguments lead to approximate solutions $(\mathfrak{u}_\epsilon^{\lambda,m}, \mathcal{A}_\epsilon^{\lambda,m})$ of the Abelian Higgs model that are described below in (92), (95), (97). The results of [18], stated in Section 5.2, establish precise estimates of the L^2 distance between these approximate solution and some actual solutions.

Next, in the same way that the formal arguments above suggest the change of variables described in Remark 1 in Section 2.3, in the vector-valued case they suggest a similar vector-valued change of coordinates which appears for example in the description (95), (97) of the approximate solution mentioned above, and also in our proofs.

Finally, the heuristic arguments are less persuasive for the Ginzburg-Landau wave equation, that is, equation (1) for $\mathbb{R}^2$-valued maps, than for the Abelian Higgs model. This reflects genuine difficulties; of all the equations we consider, (1) in the vector case $k = 2$ is the hardest to analyze, and is the one about which we obtain the weakest results. These are discussed in Section 5.1.

§3. rigorous results: interface dynamics

In this brief section we state rigorous results about the $c = 1, d = 0$ case of equation (8). The next section gives a complete proof of a simple theorem, and discusses some of the additional ingredients in the proofs of the stronger results stated here.

We consider a real-valued u solving the equation

$$(53) \qquad u_{tt} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

where $f = F'$, and F satisfies (9). We first state a typical result and then discuss some variations.

Theorem 1. *Assume that $T_* < 0 < T^*$, and that $\Gamma \subset (T_*, T^*) \times \mathbb{R}^n$ is a smooth embedded timelike hypersurface of vanishing Minkowskian mean curvature. Then given T_-, T_+ such that $T_* < T_- < 0 < T_+ < T^*$, there exists a neighborhood $\mathcal{N}$ of Γ in $(T_-, T_+) \times \mathbb{R}^n$ in which the signed distance function δ_Γ (that is, the solution of (12), (16)) is well-defined and smooth, and for every $\epsilon \in (0, 1]$, there exists a a solution u_ϵ of (53) such that*

$$\int_{[(T_-, T_+) \times \mathbb{R}^n] \setminus \mathcal{N}} \left[\chi_N \delta_\Gamma^2 + (1 - \chi_\mathcal{N}) \right] \left(u_t^2 + |\nabla u|^2 + \frac{1}{\epsilon^2} F(u) \right) dx \, dt \ \le \ C\epsilon$$

$$\int_\mathcal{N} (u_\epsilon - U_\epsilon)^2 \, dx \, dt \le C\epsilon.$$

where $U_\epsilon = q(\delta_\Gamma / \epsilon)$ and q solves (11). Also,

$$\int_{\{t\} \times \mathbb{R}^n} \left(u_t^2 + |\nabla u|^2 + \frac{1}{\epsilon^2} F(u) \right) \ge C/\epsilon$$

for every t.

This is proved in [35] with some convenient but unnatural restrictions on the topology of Γ. The arguments whereby these restrictions may be dropped are presented in [24].

Recall that the existence and smoothness, near Γ, of the signed distance function δ_Γ is standard and can be found in in Lemma 1 above.

The first estimate asserts that the energy is strongly concentrated near Γ, which is the only place at which the weight $\left[\chi_N \delta_\Gamma^2 + (1 - \chi_\mathcal{N}) \right]$ vanishes. The point of the final assertion is simply that the total energy is large, so that the first estimate is nontrivial.

Remark 2. *The theorem in fact proves not only the existence of a single solution with the stated properties, but more general stability estimates. These state that for any solution with initial data that "has an interface near Γ_0", roughly speaking, the energy concentration around Γ and gradients of the solution in directions tangent to Γ are of the same order of magnitude for $t \in (T_-, T_+)$ as they are at $t = 0$. Examples of this sort of stability estimate are presented below in Lemma 7 and (without a full proof) Lemma 9.*

Remark 3. *The presentation in* [35] *extracts more conclusions than we have stated here from the stability estimates described above. In particular these include estimates in* $W^{-1,1}$ *of the difference between the energy-momentum tensor for* u_ϵ *and the energy-momentum tensor for* Γ *(which is a tensor-valued measure concentrated on* Γ.) *See* [35] *for a precise statement.*

Remark 4. *The theorem remains true if Minkowski space is replaced by certain more general Lorentzian manifolds, see* [24].

Remark 5. *A variant of the theorem holds if the nonlinearity* $f(u)$ *is replaced by a nonlinearity* $f_\epsilon(u)$ *associated with a double-well potential with two wells of (slightly) unequal depth. The prototype is a cubic nonlinearity, say*

$$f_\epsilon(u) = (u^2 - 1)(u - \epsilon k),$$

so that $f_\epsilon = F_\epsilon'$ *with*

$$F_\epsilon = \frac{1}{4}(u^2 - 1)^2 + \epsilon k(u - \frac{1}{3}u^3)$$

so that $F_\epsilon(1) = -F_\epsilon(-1) = \frac{2}{3}\epsilon k$. *In this case, solutions have roughly the form* $u_\epsilon \approx q(\delta_\Gamma/\epsilon)$ *where* δ_Γ *is the signed distance to a hypersurface of constant Minkowskian mean curvature proportional to* k. *This too is proved in* [24] *and remains valid, with suitable modifications, in more general Lorentzian spacetimes, and also for inhomogeneous nonlinearities, whose protoype is*

$$f_\epsilon(t, x, u) = (u^2 - 1)(1 - \epsilon k(t, x)), \qquad k(\cdot, \cdot) \text{ smooth.}$$

§4. elements of proofs: interfaces in scalar equations

In this section we prove a result (Theorem 2 in Section 4.5) that illustrates some points in the proof of Theorem 1, and more generally shows how one can convert the heuristic considerations from Section 2 into rigorous arguments. We will suppress a number of difficulties by imposing strong hypotheses and proving weak conclusions. Following the proof, we discuss modifications to the basic argument that lead to more satisfying results such as those stated in Theorem 1.

As in Section 2, we consider the equation

$$(54) \qquad c^2 u_{tt} + d^2 u_t - \Delta u + \frac{1}{\epsilon^2} f(u) = 0, \qquad u : (0, T) \times \mathbb{R}^n \to \mathbb{R}$$

where $f = F'$ for F satisfying (9). We briefly discuss the $c = 0$ case in Section 4.7. Apart from that, we assume throughout this section that

$c \neq 0$. Since we allow $d \neq 0$, the setting here is a little more general than those of [35, 24], and the results Theorem 2 are new. In particular, the arguments here illustrate how essentially the same framework can be used to analyze both hyperbolic and parabolic equations, as well as equations that interpolate between them.

We will always consider smooth solutions of (54). In practice, H^2 is all the smoothness we need, and all our conclusions are true in the energy space as long as (54) is well-posed in the H^1, since then solutions with H^1 data can be approximated in H^1 by solutions for which our estimates hold. Well-posedness can be guaranteed by imposing suitable growth conditions on f.

Suppose that U_ϵ is an approximate solution as constructed in Section 2, so that

$$U_\epsilon = q(\frac{\delta_\Gamma}{\epsilon}), \quad \text{where } \delta_\Gamma \text{ solves (12), (16), (39)}$$

in a neighborhood of some hypersurface Γ. Thus Γ solves an evolution equation that may be written

(55) $c^2 a + (1 - c^2 v^2)(d^2 v - H_{\Gamma_t}^e) = 0$ (in notation from Section 2.5)

and δ_Γ is the appropriate (depending on c) signed distance from Γ. For convenience we also assume that

(56) $\partial_t \delta_\Gamma = 0$ when $t = 0$, on its domain.

This occurs if Γ has velocity zero when $t = 0$.

4.1. basic idea

The formal argument of Section 2 shows that, if one looks for a solution of the form

(57) $$U_\epsilon \approx q(\frac{\delta_\Gamma}{\epsilon})$$

then roughly the best that can be done is to take q to be the standard profile, as discussed in Section 2.2, and δ_Γ to be the signed distance from a submanifold Γ solving the appropriate geometric evolution problem (55).

In the proof, we invert the logic, to some extent, by proving that if δ_Γ is assumed from the outset to be the signed distance from a submanifold Γ solving (55), then the ansatz (57) is dynamically stable. This is proved by weighted energy estimates in a well-chosen coordinate system that follows Γ. This coordinate system is in fact exactly the one, described in Remark 1 in Section 2.3, that emerges naturally from consideration of the eikonal equation (12), (16).

4.2. change of variables

Given a subset of Γ parametrized by a map $\Psi : O \subset \mathbb{R}^n \to \mathbb{R}^{1+n}$ of the form (30), we define a map ϕ as in (32), which is a diffeomorphism once its domain is restricted to a suitable neighborhood of $O \times \{0\}$. Having done this, we can rewrite (54) in terms of coordinates $(y^0, \ldots, y^n)$ defined by $(t, x^1, \ldots, x^n) = \phi(y^0, \ldots, y^n)$. A great advantage of this procedure is that, as noted earlier, our approximate solution U_ϵ satisfies

$$U_\epsilon \circ \phi(y^0, \ldots, y^n) = q(y^n/\epsilon)$$

and so to find a solution such that $u_\epsilon \approx U_\epsilon$, it should suffice, to arrange that $v_\epsilon = u_\epsilon \circ \phi$ satisfies $v_\epsilon(0, y^1, \ldots, y^n) \approx q(\frac{y^n}{\epsilon})$, and then to show that $\partial_{y^0} v \approx 0$.

Thus, this change of variables reduces the study of the dynamics of interfaces to a question about stability of specific initial data $(v, \partial_{y^0} v) = (q(y^n/\epsilon), 0)$ when $y^0 = 0$, for a modified equation, see (58) below. The *key point* is that the transformed equation inherits certain good properties, see (59), (60) below, from the evolution equation satisfied by the surface Γ. It is precisely these good properties that make the desired stability estimates possible.

Lemma 4. *If u solves* (54) *with $c > 0$, then $v = u \circ \phi$ solves*

$$(58) \qquad -\partial_{y^\beta}(g^{\alpha\beta}\partial_{y^\alpha} v) + b^\alpha \partial_{y^\alpha} v + \frac{1}{\epsilon^2} f(v) = 0,$$

where $(g^{\alpha\beta})$ are defined in (51), *and*

$$(59) \qquad b^n = \left((c^2 \partial_{tt} + d^2 \partial_t - \Delta)\delta_\Gamma \right) \circ \phi,$$

As a result (in view of (39) *and* (33)),
(60)
$b^n(y) \leq C|y^n|, \quad$ *for C uniform on compact subsets of Domain(ϕ).*

Proof. Let us write $Y := \phi^{-1}$, so that $u = v \circ Y$ on the image of ϕ. Thus (writing $x^0 = t$)

$$\partial_{x^\alpha} u = (\partial_{y^\mu} v \circ Y) \, \partial_{x^\alpha} Y^\mu,$$

$$\partial_{x^\beta \, x^\alpha} u = (\partial_{y^\mu} v \circ Y) \, \partial_{x^\beta \, x^\alpha} Y^\mu + (\partial_{y^\nu \, y^\mu} v \circ Y) \, \partial_{x^\beta} Y^\nu \, \partial_{x^\alpha} Y^\mu.$$

So

$$\left((c^2 \partial_{tt} + d^2 \partial_t - \Delta) u \right) \circ \phi = A^{\mu\nu} \partial_{y^\mu \, y^\nu} v + B^\mu \partial_{y^\mu} v$$

for

$$A^{\mu\nu} := \left(c^2\partial_t Y^\mu \ \partial_t Y^\nu - \nabla Y^\mu \cdot \nabla Y^\nu\right) \circ \phi \ = \ -\left(\eta^{\alpha\beta}\partial_{x^\beta} Y^\nu \ \partial_{x^\alpha} Y^\mu\right) \circ \phi,$$
$$B^\mu := \left((c^2\partial_{tt} + d^2\partial_t - \Delta)Y^\mu\right) \circ \phi.$$

It follows from the definitions (51) of $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$, together with the fact that $DY \circ \phi = (D\phi)^{-1}$, that

$$A^{\mu\nu} = -g^{\mu\nu}.$$

Combining thse facts, we find that (54) transforms to

$$-\partial_{y^\mu}(g^{\mu\nu}\partial_\nu v) + b^\mu\partial_{y^\mu} v + \frac{1}{\epsilon^2}f(v) = 0$$

where

$$b^\mu := B^\mu + \partial_{y^\nu} g^{\mu\nu}.$$

To complete the proof we must verify (59). Since we have seen in (35) that $Y^n = \delta_\Gamma$, from the definitions we see that we only need to check that $\partial_{y^\nu} g^{n\nu} = 0$, and this follows immediately from Lemma 5 below.   Q.E.D.

The next Lemma completes the proof of the result above, and also will be useful in our subsequent analysis.

Lemma 5. *If $c > 0$, then inverse metric tensor satisfies*

$$g^{nn} = 1, \quad g^{an} = g^{na} = 0 \quad \text{if } a < n.$$

Moreover, on a sufficiently small neighborhood of O, $g^{00} < 0$ and $(g^{ij})_{i,j=1}^{n-1}$ is positive definite.

Proof. Let us write $\tilde{\nu} := \nu \circ \Psi$, so that $\phi = \Psi + y^n\tilde{\nu}$. Then we recall from the definitions that

$$\eta_{\alpha\beta}\tilde{\nu}^\alpha\tilde{\nu}^\beta = 1, \qquad \eta_{\alpha\beta}\tilde{\nu}^\alpha\partial_{y^a}\Psi^\beta = 0, a = 0, \ldots, n-1.$$

Differentiating the first identity, we find that $\eta_{\alpha\beta}\tilde{\nu}^\alpha\partial_{y^a}\tilde{\nu}^\beta = 0$ for $a = 0, \ldots, n-1$. Using these facts, it follows directly from the definition (51) of $(g_{\alpha\beta})$ that

(61) $$g_{nn} = 1, \qquad g_{an} = g_{na} = 0 \quad \text{if } a < n.$$

Then the same properties are inherited by $g^{n\alpha}$ for $\alpha \in \{0, \ldots, n\}$.

Next, note that due to (30) and (31), when $y^n = 0$ we have $g_{00} = -c^{-2} + v^2 < 0$ (since Γ is assumed to be timelike), $g_{0a} = g_{0a} = 0$ for $a = 1, \ldots, n-1$, and

$$(g_{ij})_{i,j=1}^{n-1} = (\partial_{y^i} \psi \cdot \partial_{y^j} \psi)_{i,j=1}^{n-1}$$

is positive definite, since ψ is by assumption nondegenerate. It follows from these and (61) that $g^{00} < 0$ and $(g^{ij})_{i,j=1}^{n-1}$ is positive definite when $y^n = 0$ (that is, at points in O), and the same facts then hold in a neighborhood of O by continuity.

<div align="right">Q.E.D.</div>

4.3.   a differential energy inequality

Next we define

(62)
$$a^{\alpha\beta} := \begin{cases} -g^{00} & \text{if } \alpha = \beta = 0 \\ g^{\alpha\beta} & \text{if } \alpha \geq 1 \text{ and } \beta \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If we restrict our attention to a domain on which the conclusions of Lemma 5 hold, then $(a^{\alpha\beta})$ is positive definite. We define the energy

$$e_\epsilon(v) := \frac{\epsilon}{2} a^{\alpha\beta} \partial_{y^\alpha} v \, \partial_{y^\beta} v + \frac{1}{\epsilon} F(v).$$

Lemma 6. *If v is a smooth solution of* (58), (60), *then*

(63) $$\frac{\partial}{\partial y^0} e_\epsilon(v) \leq \epsilon \partial_{y^i} (g^{i\alpha} \partial_{y^\alpha} v \, \partial_{y^0} v) + C\epsilon \left(|D_\tau v|^2 + (y^n)^2 (\partial_{y^n} v)^2 \right),$$

where $|D_\tau v|^2 := \sum_{a=0}^{n-1} (\partial_{y^a})^2$, and C is uniform on compact subsets of Domain(ϕ).

In (63), and throughout this paper, we implicitly sum roman indices i, j, k from 1 to n, except where specified otherwise. We recall that α, β, μ, ν are summed from 0 to n.

Proof.   Multiply (58) by $\partial_{y^0} v$ and rearrange to find that

$$-\partial_{y^\beta} (g^{\alpha\beta} \partial_{y^\alpha} v \, \partial_{y^0} v) + g^{\alpha\beta} \partial_{y^\alpha} v \, \partial_{y^\beta y^0} v + \partial_{y^0} F(v) = b^\alpha \, \partial_{y^\alpha} v \, \partial_{y^0} v.$$

We can rewrite

$$g^{\alpha\beta} \partial_{y^\alpha} v \, \partial_{y^\beta y^0} v = \frac{1}{2} \partial_{y^0} (g^{\alpha\beta} \partial_{y^\alpha} v \, \partial_{y^\beta} v) - \frac{1}{2} (\partial_{y^0} g^{\alpha\beta}) \partial_{y^\alpha} v \, \partial_{y^\beta} v.$$

Combining these identities and rearranging,

$$\partial_{y^0} \left[\frac{1}{2} g^{\alpha\beta} \partial_{y^\alpha} v\, \partial_{y^\beta} v - g^{\alpha 0} \partial_{y^\alpha} v\, \partial_{y^0} v + F(v) \right]$$

$$= \partial_{y^i} (g^{i\alpha} \partial_{y^\alpha} v\, \partial_{y^0} v) + \frac{1}{2} (\partial_{y^0} g^{\alpha\beta}) \partial_{y^\alpha} v\, \partial_{y^\beta} v + b^\alpha\, \partial_{y^\alpha} v\, \partial_{y^0} v.$$

The right-hand side is just $\epsilon^{-1} \partial_{y^0} e_\epsilon(v)$, and it follows by elementary estimates from Lemma 5 and (60) that the right-hand side is bounded by

$$\partial_{y^i} (g^{i\alpha} \partial_{y^\alpha} v\, \partial_{y^0} v) + C \left(|D_\tau v|^2 + (y^n)^2 (\partial_{y^n} v)^2 \right),$$

for C uniform on compact subsets of Domain(ϕ). Q.E.D.

4.4. weighted energy estimates for v, easiest nontrivial case.

Now we suppose for simplicity that Γ is parametrized by a map $\Psi : [0, T] \times \mathbb{T}^{n-1} \to \mathbb{R}^{1+n}$ of the form (30), where $\mathbb{T}^k$ denotes the k-dimensional torus. Thus $\Psi(y^0, \ldots, y^{n-1}) = (y^0, \psi(y^0, \ldots, y^{n-1}))$, where ψ is periodic (say with period 1) in the y^i variables, $i = 1, \ldots, n-1$. This assumption imposes topological constraints on Γ that are reasonable if $n = 2$ and otherwise rather artificial.

In this case, given any $T_0 < T$, we can find some $r > 0$ (depending on T_0) such that the conclusions of Lemmas 5 and 6 hold on $M_{T_0}^r = [0, T_0] \times \mathbb{T}^{n-1} \times [-r, r]$ with uniform constants, and in addition the restricion of ϕ to $M_{T_0}^r$ is a diffeomorphism onto its image.

Lemma 7. *Let v be a smooth solution of* (58) *on $M_{T_0}^r$, for T_0 and r as above, and assume that there exists some $T_1 \in (0, T_0]$ such that*

$$(64) \qquad v(y^0, \ldots, y^n) = \text{sign}(y^n) \quad \text{if } 0 \le y^0 \le T_1 \text{ and } |y^n| = r.$$

For $s \in [0, T_0)$, define (recalling the definition (19) *of c_0)*

$$\zeta_1(s) := \int_{\{s\} \times \mathbb{T}^{n-1}} \left(\int_{[-r,r]} (1 + (y^n)^2) e_\epsilon(v) dy^n - c_0 \right) dy^1 \cdots dy^{n-1},$$

$$\zeta_2(s) := \int_{\{s\} \times \mathbb{T}^{n-1} \times [-r,r]} \epsilon |D_\tau v|^2 + (y^n)^2 \left(\epsilon(\partial_{y^n} v)^2 + \frac{1}{\epsilon} F(v) \right) dy^1 \cdots dy^n.$$

Then for $0 \le s \le T_1$,

$$\zeta_2(s) \le C\zeta_1(s) \qquad \text{and} \quad \zeta_1(s) \le e^{Cs} \zeta_1(0).$$

The assumption (64) makes this situation particularly simple. We will see below that it is easy to find initial data $u(0, \cdot), \partial_t u(0, \cdot)$ for (8) such that the corresponding solution $v = u \circ \phi$ of (58) satisfies (64), with T_1 independent of $\epsilon \in (0, 1]$. The lemma thus will yield a form of the heuristic statement "$U_\epsilon \approx u_\epsilon$" when applied to suitable initial data. However, the pointwise control required in (64) has distinct drawbacks, as we will see.

Proof. **Step 1**. We first claim that for every $(y^0, \ldots, y^{n-1})$ such that $0 \leq y^0 < T_1$,

(65)
$$\int_{-r}^{r} \frac{\epsilon}{2} (\partial_{y^n} v)^2 + \frac{1}{\epsilon} F(v) \, dy^n \Big|_{(y^0, \ldots, y^{n-1})} \geq c_0.$$

Indeed, this follows from integrating the inequality

$$\sqrt{2F(v)} \partial_{y^n} v \leq \frac{\epsilon}{2} (\partial_{y^n} v)^2 + \frac{1}{\epsilon} F(v)$$

from $y^n = -r$ to r and using (64) and the definition (19) of c_0.

Step 2. By Lemma 5,

$$e_\epsilon(v) \geq c\epsilon |D_\tau v|^2 + \frac{\epsilon}{2} (\partial_{y^n} v)^2 + \frac{1}{\epsilon} F(v).$$

So by fixing $s \in [0, T_1)$ and integrating (65) over $(y^1, \ldots, y^{n-1}) \in \mathbb{T}^{n-1}$ with $y^0 = s$, we deduce that

(66)
$$\int_{\{s\} \times \mathbb{T}^{n-1}} \left(\int_{[-r,r]} e_\epsilon(v) dy^n - c_0 \right) dy^1 \cdots dy^{n-1}$$

$$\geq \int_{\{s\} \times \mathbb{T}^{n-1} \times [-r,r]} c\epsilon |D_\tau v|^2.$$

It immediately follows that $\zeta_2(s) \leq C\zeta_1(s)$ for $0 \leq s < T_1$.

Step 3. Next we claim that

(67)
$$\zeta_1'(s) \leq C\zeta_2(s) \qquad \text{for} \quad s \in [0, T_1].$$

In view of Step 2, this will prove that $\zeta_1' \leq C\zeta_1$, and then the remaining conclusion of the Lemma will follows from Grönwall's inequality.

To prove (67), we employ the differential energy inequality (63) to find that

$$\zeta_1'(s) = \int_{\{s\} \times \mathbb{T}^{n-1} \times [-r,r]} (1 + (y^n)^2) \, \partial_{y^0} e_\epsilon(v) \, dy^1 \cdots dy^n$$

$$\leq \int_{\{s\} \times \mathbb{T}^{n-1} \times [-r,r]} \epsilon (1 + (y^n)^2) \partial_{y^i} (g^{i\alpha} \partial_{y^\alpha} v \; \partial_{y^0} v) dy^1 \cdots dy^n + C\zeta_2(s).$$

Now we integrate by parts and recall from Lemma 5 properties of $(g^{n\alpha})$ to find that

$$\zeta_1'(s) \le 2\epsilon \int_{\{s\}\times\mathbb{T}^{n-1}\times[-r,r]} |y^n|\,|\partial_{y^n} v|\,|\partial_{y^0} v|\,dy^1 \cdots dy^n \;+\; C\zeta_2(s).$$

There are no boundary terms coming from $y^1, \ldots, y^{n-1}$ by periodicity, and none from y^n since $\partial_{y^0} v = 0$ when $|y^n| = r$, by (64). Since

$$|y^n|\,|\partial_{y^n} v|\,|\partial_{y^0} v| \le \frac{1}{2}(y^n)^2(\partial_{y^n} v)^2 + \frac{1}{2}|D_\tau v|^2,$$

it follows that $\zeta_1' \le C\zeta_2$ for $s \le T_1$. $\hfill$ Q.E.D.

4.5. a sample theorem

In this section we complete the proof of a relatively simple specific instance of the heuristic principle $u_\epsilon \approx U_\epsilon$.

We continue to assume that the hypersurface Γ and numbers T_0, r have the properties described at the beginning of Section 4.4. These assumptions imply that Γ is the boundary in $[0,T) \times \mathbb{R}^n$ of a bounded (relatively) open set, say $\mathcal{O}$. We may choose the Minkowskian unit normal ν to point into $\mathcal{O}$, so that $\delta_\Gamma(t,x) > 0$ for $(t,x) \in \mathcal{O}\cap\mathrm{Domain}(\delta_\Gamma)$.

We will write $\mathcal{O}_t := \{x \in \mathbb{R}^n : (t,x) \in \mathcal{O}\}$, so that Γ_t is the boundary in $\mathbb{R}^n$ of $\mathcal{O}_t$. If S is any set, we will write χ_S for the characteristic function of S, and we use the notation

$$\mathrm{sign}_S := \begin{cases} 1 & \text{if } p \in S \\ -1 & \text{if not.} \end{cases}.$$

Again, for the sake of simplicity, we have imposed a very strong pointwise hypothesis, see (68).

Theorem 2. *Assume that u is a smooth solution of* (8) *with initial data such that*

(68)
$$u_0(x) := u(0,x) = \mathrm{sign}_{\mathcal{O}_0}(x) \quad \text{if } \mathrm{dist}(x,\Gamma_0) \ge \frac{r}{2},$$
$$\partial_t u(0,x) = 0 \qquad\qquad everywhere.$$

Then there exists some $T_2 > 0$, independent of $\epsilon \in (0,1]$, and an open neighborhood $\mathcal{N}$ of Γ in $[0,T_2] \times \mathbb{R}^n$ such that

(69)
$$u = \mathrm{sign}_{\mathcal{O}} \quad in \quad ([0,T_2] \times \mathbb{R}^n) \setminus \mathcal{N},$$

and

(70)
$$\int_{\mathcal{N}} |u - U_\epsilon|^2 dt\,dx \;\le\; \frac{C}{\epsilon}\zeta_0 + \int_{\{x\in\mathbb{R}^N:\mathrm{dist}(x,\Gamma_0)<\frac{r}{2}\}} |u_0 - U_\epsilon(0,\cdot)|^2.$$

Here $\zeta_0 := \zeta_1(0)$ as defined in Lemma 7, and v appearing in the definition of $\zeta_1(s)$ is given by $v = u \circ \phi$.

An appropriate choice of initial data will then yield the following.

Corollary 1. *Under the above assumptions, for every $\epsilon \in (0, 1]$ there exists a solution u_ϵ of (8) such that (69) holds and*

$$\int_{\mathcal{N}} |u - U_\epsilon|^2 dt \ dx \ \leq C\epsilon.$$

The proof of Theorem 2 is conceptually very simple. First, using assumption (68) about the initial data and standard considerations involving finite propagation speed for semilinear wave equations, it is easy to prove that (69) holds for a suitable choice of $\mathcal{N}$.

Then from (69), we deduce that $v = u \circ \phi$ satisfies (64) for some time $T_1 > 0$. We may thus apply Lemma 7 to obtain, among other things, L^2 estimates of $\partial_{y^0} v$, and we can use these and a form of Poincarés inequality (in the y^0 variable) to control $\|v(s, \cdot) - v(0, \cdot)\|_{L^2}$ for $0 \leq s \leq T_1$. Translated back to the (t, x) variables, this will yield (70).

Proof of Theorem 2. **Step 1**. Let us write

$$e_\epsilon(u; \eta) := \frac{\epsilon}{2} \left((c\partial_t u)^2 + |\nabla u|^2 \right) + \frac{1}{\epsilon} F(u).$$

If u solves (8), then a short computation shows that

(71) $$\frac{\partial}{\partial t} e_\epsilon(u; \eta) \ = \ -\epsilon d^2 (\partial_t u)^2 + \epsilon \nabla \cdot (\partial_t u \nabla u) \ \leq \ \epsilon \nabla \cdot (\partial_t u \nabla u).$$

Then for $x_0 \in \mathbb{R}^n$ and $\rho > 0$, if $0 < t < c\rho$ we compute

$$\frac{d}{dt} \int_{B(x_0, \rho - t/c)} e_\epsilon(u; \eta)$$

$$= \int_{B(x_0, \rho - t/c)} \partial_t e_\epsilon(u; \eta) - \frac{1}{c} \int_{\partial B(x_0, \rho - t/c)} e_\epsilon(u; \eta).$$

But by (71) and the divergence theorem,

$$\int_{B(x_0, \rho - t/c)} \partial_t e_\epsilon(u; \eta) \ \leq \ \epsilon \int_{\partial B(x_0, \rho - t/c)} |\partial_t| \ |u \nabla u|.$$

Since

$$\epsilon |\partial_t| \ |u \nabla u| \leq \epsilon \left(\frac{c}{2} (u_t)^2 + \frac{1}{2c} |\nabla u|^2 \right) \leq \frac{1}{c} e_\epsilon(u; \eta)$$

we conclude that $\frac{d}{dt} \int_{B(x_0, \rho - t/c)} e_\epsilon(u; \eta) \leq 0$.

As a result, if $B(x_0, \rho) \subset \mathbb{R}^n$ is any ball on which $e_\epsilon(u; \eta)$ vanishes when $t = 0$, then for $0 < t < c\rho$, the same quantity also vanishes on $B(x_0, \rho - t/c)$, and hence u is constant on the cone $\cup_{0 < t < c\rho} \{t\} \times B(x_0, \rho - t/c)$. It follows from this and (68) that

$$(72) \qquad u = \mathrm{sign}_{\mathcal{O}} \quad \text{on } A := \left\{ (t, x) : t \geq 0, \mathrm{dist}(x, \Gamma_0) > \frac{r}{2} + \frac{t}{c} \right\},$$

where dist denotes the Euclidean distance.

Step 2. Our standing assumption (56) that $\partial_t \delta_\Gamma = 0$ when $t = 0$ implies that the Minkowskian normal ν to Γ when $t = 0$ has the form $\nu = (0, \nu^e)$, where $\nu^e \in \mathbb{R}^n$ is the Euclidean unit normal to Γ_0. It follows that $\phi(0, y^1, \ldots, y^n)$ has the form $(0, \phi_0(y^1, \ldots, y^n))$, where

$$(73) \qquad \phi_0(y^1 \ldots, y^n) = \psi_0(y^1, \ldots, y^{n-1}) + y^n \nu^e(\psi_0(y^1, \ldots, y^{n-1}),$$

and $\psi_0(\cdot) := \psi(0, \cdot)$.

We deduce from (73) that our change of variables maps the Cauchy problem for u onto a standard Cauchy problem for v rather than, say, a problem for which the initial data is given on some hypersurface not of the form $\{y^0 = \mathrm{const}\}$. This is the simplification that we gain from assumption (56).

Step 3. We next claim that there exists $T_1 > 0$ such that $v = u \circ \phi$ satisfies the main hypothesis (64) of Lemma 7, which we recall is that

$$v(y^0, \ldots, y^n) = \mathrm{sign}(y^n) \text{ if } 0 \leq y^0 \leq T_1 \text{ and } |y^n| = r.$$

We have arranged that $\mathrm{sign}\, y^n = \mathrm{sign}(\delta_\Gamma \circ \phi(y)) = \mathrm{sign}_{\mathcal{O}}(\phi(y))$, so it suffices by (72) to show that there exists $T^1 > 0$ such that

$$\phi(y^0, \ldots, y^n) \in A \text{ if } 0 \leq y^0 \leq T_1 \text{ and } |y^n| = r.$$

And this is clear, since (73) implies that

$$\phi\left(\{0\} \times \mathbb{T}^n \times \{\pm r\}\right) = \{(0, x) : \mathrm{dist}(x, \Gamma_0) = r\}$$

and hence this set is a compact subset of the open set A.

Step 4. We now apply Lemma 7 to conclude for example that

$$\int_{[0, T_1] \times \mathbb{T}^{n-1} \times [-r, r]} |D_\tau v|^2 \leq \frac{C}{\epsilon} \zeta_0.$$

Since $|\partial_{y^0} v| \leq |D_\tau v|$, it follows from this and a form of Poincarés inequality that (writing $v_0(y^0, y^1, \ldots, y^n) = v(0, y^1, \ldots, y^n)$)

$$(74) \qquad \int_{[0, T_1] \times \mathbb{T}^{n-1} \times [-r, r]} (v - v_0)^2 \leq \frac{C(T_1)^2}{\epsilon} \zeta_0.$$

Recall that $\phi : [0, T_0] \times \mathbb{T}^{n-1} \times [-r, r]$ is a diffeomorphism onto its image. The same thus holds for ϕ_0 on $\mathbb{T}^{n-1} \times [-r, r]$. As a result, the Jacobian determinants of ϕ, ϕ_0, and their inverses are uniformly bounded in the relevant domains, and we can change variables freely, at the expense of increasing our constants somewhat. Thus, defining $\mathcal{N}_1 := \phi([0, T_1) \times \mathbb{T}^n \times (-r, r))$, since $(u - U_\epsilon) \circ \phi = v - q(\frac{y^n}{\epsilon})$

$$\int_{\mathcal{N}_1} |u - U_\epsilon|^2 \leq C \int_{[0,T_1) \times \mathbb{T}^n \times [-r,r]} |v - q(\frac{y^n}{\epsilon})|^2$$

$$\leq C \int_{[0,T_1) \times \mathbb{T}^n \times [-r,r]} |v - v_0|^2 + |v_0 - q(\frac{y^n}{\epsilon})|^2$$

$$(75) \qquad \leq \frac{C T_1^2}{\epsilon} \zeta_0 + C T_1 \int_{\{x \in \mathbb{R}^N : \mathrm{dist}(x, \Gamma_0) < \frac{r}{2}\}} |u_0 - U_\epsilon(0, \cdot)|^2.$$

In the last line we used (74) and changed variables again.

Now we complete the proof of the theorem by remarking that there exists $T_2 > 0$ such that

$$[0, T_2] \times \mathbb{R}^n \subset \mathcal{N} \cup \bar{A}, \qquad \mathcal{N} := \mathcal{N}_1 \cap ([0, T_1] \times \mathbb{R}^n).$$

This is easy to verify. Once this holds, (69) follows from (72), and (70) is a consequence of (75). Q.E.D.

We close this section with the

Proof of Corollary 1. We define

$$u_0(x) := \begin{cases} \mathrm{sign}_{\mathcal{O}} & \text{if } \mathrm{dist}(x, \Gamma_0) \geq \frac{r}{2} \\ \tilde{q}_\epsilon(\delta_\Gamma(x)) & \text{if } \mathrm{dist}(x, \Gamma_0) < \frac{r}{2} \end{cases}$$

where

$$\tilde{q}(s) := \chi(s) \, \mathrm{sign}(s) + (1 - \chi(s)) q(\frac{s}{\epsilon})$$

and $\chi \in C_c^\infty(\mathbb{R})$ is an even function, nonincreasing on $[0, \infty)$, such that

$$(76) \qquad \chi(s) = 1 \text{ if } s \leq \frac{r}{4}, \qquad \chi(s) = 0 \text{ if } s \geq \frac{r}{3}.$$

The exponential decay (18) of q implies that $\tilde{q}(s) - q(\frac{s}{\epsilon})$ is exponentially small in ϵ (pointwise, and in any Sobolev norm), and it follows in particular that $\|u_0 - U_\epsilon(0, \cdot)\|_{L^2(\{\mathrm{dist}(\cdot, \Gamma_0) < r/2\})} \leq C e^{-c/\epsilon}$. Also, for this data, $v_0 = u_0 \circ \phi = \tilde{q}_\epsilon(y^n)$, and then it is straightforward to verify that $\zeta_0 \approx C \epsilon^2$. Q.E.D.

4.6. improvements

Theorem 2 has some of serious weaknesses:

- It requires extremely strong pointwise assumptions (68) on the initial data away from Γ_0
- Its conclusions are valid only for a short time T_1. In particular, if T denotes the first time (possibly $+\infty$) at which the submanifold Γ develops a singularity, then a better result would be one that is valid up to time T.
- It only applies to surfaces Γ such that Γ_t is homeomorphic to the $(n-1)$-torus for every t.
- It only applies to surfaces Γ whose initial velocity vanishes.

Concerning the topology of Γ, note that by assuming that Γ_t is topologically a torus for every t, we have arranged that it admits a nice parametrization by a a single coordinate chart. This is convenient, but it is certainly possible to carry out a similar argument with Γ given as the image of by a map $\Psi : [0, T) \times M \to \mathbb{R}^{1+n}$ for some arbitrary smooth $(n-1)$-manifold M, and with ϕ defined on the product manifold $[0, T) \times M \times (-r, r)$ for some r. This is carried out in detail in [24] and does not present any conceptual difficulties.

We will not say much here about the question of nonzero initial velocity, which however is considered in detail in [35].

The first two weaknesses mentioned above both arise from the same source: they are needed to guarantee that the very strong pointwise boundary conditions (64) assumed in Lemma 7 are satisfied. To relax this assumption, we recall its role in the proof of Lemma 7, which is:

$$\text{assumption (64)} \quad \Longrightarrow \quad \text{lower energy bounds} \quad \Longrightarrow \quad \zeta_2(s) \leq C\zeta_1(s)$$

where ζ_1, ζ_2 are defined in the statement of the Lemma. In particular, lower energy bounds (65) that follow from (64) are used to neutralize a dangerous negative term in ζ_1 and hence show that it possesses some coercivity properties. Without such lower bounds, control of ζ_1 can yield no useful information.

We therefore need to identify weaker assumptions under which lower bounds like those in the proof of Lemma 7 still hold. This is done in the following

Lemma 8. *For every $r > 0$, there exists a constants $\kappa_1, C > 0$ such that if $v :\in H^1([-r, r])$ satisfies*

$$(77) \qquad\qquad \int_{-r/2}^{r/2} |s|\, |v(s) - \mathrm{sign}(s)|^2 \, ds \leq \kappa_1$$

then

$$\int_{r/2}^{r/2} \frac{\epsilon}{2}(v')^2 + \frac{1}{\epsilon}F(v) \; ds \geq c_0 - Ce^{-C/\epsilon}.$$

The proof, which is not very difficult, is given in [35], Lemma 11. The informal idea is that if κ_1 is fixed to be small enough, then

$$(77) \quad \Longrightarrow \quad \text{``}v \text{ has an interface near } s = 0\text{''},$$

and this implies lower energy bounds.

Using Lemma 8 one can prove

Lemma 9. *Let v be a smooth solution of* (58) *on $M_{T_0}^r$, for T_0 and r as in Lemma 7.*

For a constant c_* to be specified later, and depending on Γ, r, T_0, for $s < r/c_*$ let $I(s)$ denote the interval $(-r + c_* s, r - c_* s)$, and define

$$\zeta_1(s;\tau) := \int_{\{s+\tau\}\times\mathbb{T}^{n-1}} \left(\int_{I(s)} (1 + (y^n)^2) e_\epsilon(v) dy^n - c_0 \right) dy^1 \cdots dy^{n-1},$$

$$\zeta_2(s;\tau) := \int_{\{s+\tau\}\times\mathbb{T}^{n-1}\times I(s)} \epsilon |D_\tau v|^2 + (y^n)^2 \left(\epsilon(\partial_{y^n} v)^2 + \frac{1}{\epsilon}F(v) \right) dy^1 \cdots dy^n$$

$$\zeta_3(s;\tau) := \int_{\{s+\tau\}\times\mathbb{T}^{n-1}\times[-\frac{r}{2},\frac{r}{2}]} |y^n| \; |v - \text{sign}(y^n)|^2 dy^1 \cdots dy^n.$$

Then for any $\tau < T_0$ and for $0 \leq s \leq T_1 = \max(T_0 - \tau, \frac{r}{2c_*})$:

$$\zeta_2(s;\tau) \leq C_1 \zeta_1(s;\tau) + C_3 \zeta_3(s;\tau) + Ce^{-C/\epsilon}, \qquad \partial_s \zeta_1(s;\tau) \leq C\zeta_2(s;\tau)$$

and

$$\zeta_3(s;\tau) \leq 2\zeta_3(\tau) + C \int_0^s \zeta_2(\sigma;\tau) \; d\sigma.$$

for constants independent of ϵ.

Using the Lemma, one can easily argue by Grönwall's inequality that $\max_{i=1,2,3} \zeta_i(s;\tau) \leq Ce^{Cs} \max_{j=1,2} \zeta_j(0;\tau)$.

It is easy to convince oneself that Lemma 9 should allow for an improvement of Theorem 2 that that does not require pointwise assumptions about the initial data, replacing these assumptions by integral estimates, including smallness of $\zeta_3(0;0)$.

It is a little less obvious, but also true, that one can use Lemma 9 to prove that conclusions like those of Theorem 2 hold up to any time $T_0 < T$, where T is the first time at which Γ develops a singularity. This can be done by an iterative argument that switches back and forth between

energy estimates for u in the (t, x) variables and for v in the y variables, where the latter just consist in invoking Lemma 9 for successively larger values of τ. At each step, smallness conditions for either u or v are verified by using previous conditions proved previously for *both u and v*. It is proved in [35] that by arguing in this way, one can reach any time $T_0 < T$.

We sketch some elements of the proof of Lemma 9:

- The estimate of ζ_2, which proves the coercivity of ζ_1 (modulo errors controlled by ζ_3) rests on good lower energy bounds that are deduced from Lemma 8, together with an argument involving Chebyshev's inequality that uses $\zeta_3(s)$ to control the size of the set of points $(y^1, \ldots, y^{n-1}) \in \mathbb{T}^{n-1}$ on which the map $z \mapsto v(s, y^0, \ldots, y^{n-1}, z)$ fails to satisfy hypothesis (77).
- To estimate ζ_1', as before we differentiate, appeal to the differerential energy inequality (63), and integrate by parts. The boundary terms arising from the integration by parts, which earlier vanished due to the pointwise assumption (64), here are controlled by taking the constant c_* sufficiently large.
- To estimate ζ_3, we fix $Q : \mathbb{R} \to \mathbb{R}$ such that $Q' = \sqrt{2F}$, and we note that

$$|y^n| \, \frac{\partial}{\partial y^0} Q(v) \;=\; |y^n| \sqrt{2F(v)} \, \partial_{y^0} v \leq C \left(\epsilon |D_\tau v|^2 + \frac{(y^n)^2}{\epsilon} F(v) \right).$$

  By integrating this identity we find that changes in ζ_3 are controlled by integrals of ζ_2.

4.7.   the case $c = 0$

In this section we present a very sketchy discussion of the parabolic case $c = 0$. In this case, we can no longer take advantage of finite propagation speed. On the other hand, things simplify somewhat in that the diffeomorphism $\phi = (\phi^0, \ldots, \phi^n)$ defined in (32) now preserves the time variable.

The argument we sketch here is conceptually very similar to the parabolic weighted energy estimates introduced in [62] and discussed in the introduction, and as mentioned there, there are much stronger results available concerning this equation.

We proceed as follows, setting $d = 1$ in (8) for simplicity. The change of variables $v = u \circ \phi$ then leads to the equation

$$\partial_{y^0} v - \partial_{y^i}(g^{ij} \partial_{y^j} v) + b^i \partial_{y^i} v + \frac{1}{\epsilon^2} f(v) = 0$$

where i, j are summed implictly from 1 to n, and

$$b^n = (\partial_t - \Delta)\delta_\Gamma = O(|y^n|)$$

using properties of the change of variables and the equation solved by Γ. In this case, (g^{ij}) is positive definite, and the natural energy is simply

$$e_\epsilon(v) := \frac{\epsilon}{2}g^{ij}\partial_{y^i}v\,\partial_{y^j}v + \frac{1}{\epsilon}F(v).$$

Also, it remains true that $g^{in} = g^{ni} = \delta^{in}$ for $i \geq 1$. Parallel to (63), we have a differential energy inequality

$$\partial_{y^0}e_\epsilon(v) \leq \epsilon\partial_{y^i}(g^{ij}\partial_{g^j}v\,\partial_{y^0}v) - \frac{\epsilon}{2}(\partial_{y^0}v)^2 + C\left(\epsilon|\nabla_\tau v|^2 + (y^n)^2(\partial_{y^n}v)^2\right)$$

where $|\nabla_\tau v|^2 = \sum_{i=1}^{n-1}(\partial_{y^i}v)^2$. We define χ as in (76) and

$$\zeta_1(s) := \int_{\{s\}\times\mathbb{T}^{n-1}}\left(\int_{[-r,r]}\chi(y^n)(1+(y^n)^2)e_\epsilon(v)dy^n - c_0\right)dy^1\cdots dy^{n-1},$$

$$\zeta_2(s) := \int_{\{s\}\times\mathbb{T}^{n-1}\times[-r,r]}\chi(y^n)\left[\epsilon|\nabla_\tau v|^2 + (y^n)^2\left(\epsilon(\partial_{y^n}v)^2 + \frac{1}{\epsilon}F(v)\right)\right]dy^1\cdots dy^n,$$

$$\zeta_3(s) := \int_{\{s\}\times\mathbb{T}^{n-1}\times[-\frac{r}{4},\frac{r}{4}]}|y^n|\,|v - \text{sign}(y^n)|^2 dy^1\cdots dy^n,$$

$$\zeta_4(s) := \int_{\{s\}\times\mathbb{R}^n}(1-\chi(\delta_\Gamma))\left[\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}F(u)\right]dx.$$

Then, parallel to Lemma 9, one can verify that

$$\zeta_1'(s) + c\epsilon\int_{\{s\}\times\mathbb{T}^n\times[-r,r]}\chi(y^n)(\partial_{y^0}v)^2 dy^1\cdots dy^n \leq C(\zeta_2(s) + \zeta_4(s)),$$

$$\zeta_2(s) \leq C_1\zeta_1(s) + C_3\zeta_3(s) + Ce^{-C/\epsilon},$$

$$\zeta_3(s) \leq 2\zeta_3(0) + C\int_0^s \zeta_2(\sigma)d\sigma,$$

$$\zeta_4'(s) + c\epsilon\int_{\{s\}\times\mathbb{R}^n}(1-\chi(\delta_\Gamma))\,(\partial_t u)^2 dx \leq C(\zeta_2(s) + \zeta_4(s)).$$

Indeed, the estimates of ζ_2, ζ_3 are exactly as in Lemma 9, and the estimates of ζ_1 follows by arguing as in the earlier lemma, but keeping track of an additional negative term and using $\zeta_4(s)$ to help control the boundary terms arising from integration by parts. It is at this point (and in the estimate of ζ_4, which is very similar to that of ζ_1) that we use the fact ϕ preserves the time variable.

Once the above analog of Lemma 9 is established, one can conclude (by some argument involving Grönwall's inequality, temporarily ignoring the $(\partial_{y^0} v)^2$ and $(\partial_t u)^2$ terms in the first and fourth inequalities) that

$$\max_{i=1,2,3,4} \zeta_i(s) \le Ce^s \max_{j=1,2,3} \zeta_j(0) + Ce^{-C/ep}.$$

Then recalling for example the $(\partial_{y^0} v)^2$, term we find that

$$\int_0^s c\epsilon \int_{\{s\}\times\mathbb{T}^n\times[-r,r]} \chi(y^n)(\partial_{y^0} v)^2 dy^1 \cdots dy^n \ \le \ \frac{C}{\epsilon} e^s \max_{j=1,2,3} \zeta_j(0)$$

As before, one can find data such that $\zeta_j(0) \le C\epsilon^2$ for $j = 1, 2, 3$ and then prove via Poincaré's inequality a statement of the form

(78)
$$\int_{\mathcal{N}} |u_\epsilon - U_\epsilon|^2 \le C\epsilon$$

for a suitable neighborhood of $\mathcal{N}$ of Γ in $[0, T_0] \times \mathbb{R}^n$ for any T_0 less than the first time T at which Γ develops singularities.

The above estimates also imply other conclusions such as energy concentration around Γ, as in the original weighted energy estimates of Soner [62]. The L^2 estimate (78), however, is not established in [62] and would require some work to extract from the basic estimates proved there.

Remark 6. *When $c \ne 0$ one could presumably foliate a subset of $\mathbb{R} \times \mathbb{R}^n$ that contains $[0, T_0] \times \mathbb{R}^n$ by spacelike hypersurfaces Σ_t such that*

- *$\{t\} \times \Gamma_t \subset \Sigma_t$, and*
- *Γ is normal to Σ_t at every point of $\{t\} \times \Gamma_t$.*

This done, one could argue as above with $\zeta_1, \ldots, \zeta_4$, with no need to switch back and forth iteratively between (t, x) and y variables. (In our sketch of the parabolic argument, a limited amount of switching back and forth is needed in the estimates of ζ_1' and ζ_4', but it is only needed once in each estimate.)

§5. rigorous results: vortex/string dynamics

In this section we state some rigorous results about dynamics of energy concentration sets in solutions of complex-valued semilinear wave equations. These theorems assert the existence of solutions for which energy concentrates around a *codimension 2* submanifolds, which are sometimes referred to in the literature as either strings or vortices. We will discuss some aspects of the proofs in the following section.

5.1. the Ginzburg-Landau wave equation

We first consider the simplest vector-valued analog of the equation treated in Theorem 1:

$$(79) \qquad u_{tt} - \Delta u + \frac{1}{\epsilon^2} f(|u|^2) u = 0, \qquad u : \mathbb{R}^{1+n} \to \mathbb{R}^2$$

where $f : [0, \infty) \to \mathbb{R}$ is a strictly increasing function such that $f'(0) = 0$ and $f(1) = 0$. We will write F to denote the function such that

$$f = 2F', \qquad F(1) = 0, \qquad F(s) > 0 \text{ for } s \neq 1.$$

The prototype is $f(|u|^2) = |u|^2 - 1$ and $F(|u|^2) = \frac{1}{4}(|u|^2 - 1)^2$.

There is no signed distance function for submanifolds of codimension > 1, so here we write δ_Γ to denote the (unsigned) distance function to an extremal codimension 2 timelike submanifold, characterized in a neighborhood $\mathcal{N}$ of a submanifold Γ by

$$\delta_\Gamma = 0 \text{ on } \Gamma,$$

and

$$\delta_\Gamma > 0, \qquad \eta^{\alpha\beta} \partial_{x^\alpha} \delta_\Gamma \, \partial_{x^\beta} \delta_\Gamma = 1 \text{ in } \mathcal{N} \setminus \Gamma.$$

This distance function is smooth in $\mathcal{N} \setminus \Gamma$, for suitable $\mathcal{N}$, but not at points of Γ.

Theorem 3. *Assume that $T_* < 0 < T^*$, and that $\Gamma \subset (T_*, T^*) \times \mathbb{R}^n$ is a smooth embedded timelike* codimension 2 *surface of vanishing mean curvature. Assume further that Γ_t is diffeomorphic to $\mathbb{T}^{n-2}$ for every t.*

Then given T_-, T_+ such that $T_* < T_- < 0 < T_+ < T^*$, there exists a neighborhood $\mathcal{N}$ of Γ in $(T_-, T_+) \times \mathbb{R}^n$ in which the un*signed distance function δ_Γ is well-defined, and for every $\epsilon \in (0, 1]$, there exists a a solution u_ϵ of* (53) *such that*

$$\int_{[(T_-, T_+) \times \mathbb{R}^n] \setminus \mathcal{N}} \left[\chi_N \delta_\Gamma^2 + (1 - \chi_\mathcal{N}) \right] \left(u_t^2 + |\nabla u|^2 + \frac{1}{\epsilon^2} F(|u|^2) \right) dx \, dt \leq C,$$

and

$$\int_{\{t\} \times \mathbb{R}^n} \left(u_t^2 + |\nabla u|^2 + \frac{1}{\epsilon^2} F(|u|) \right) \geq C |\ln \epsilon|$$

for every $t \in \mathbb{R}$.

This is proved in [35], where analogs the results discussed in Remark 2 (more general stability estimates) and Remark 3 (estimates of the difference, in weak norms, between the energy-momentum tensors of u and of Γ) are also established.

Remark 7. *The proof also shows that the solution u exhibits a "vortex structure" on most cross-sections to Γ. To state this more precisely, fix $p \in \Gamma \cap \{(t,x) : T_- < t < T_+\}$, and let ν^1, ν^2 be an orthonormal basis for $(T_p\Gamma)^\perp$, where "orthonormal" and "$\perp$" are understood with respect to the Minkowski metric. We then define*

$$(80) \qquad u_p(z^1, z^2) := u(p + z^1\nu^1 + z^2\nu^2),$$

for $z = (z^1, z^2)$ in a small ball around the origin in $\mathbb{R}^2$. The proof shows that, except for points p in a subset of Γ of small measure,

$$(81) \qquad ||| \, \omega(u_p) - \pi\delta_0 \, ||| \ll 1$$

where $|||\cdot|||$ denotes a weighted $W^{-1,1}$ norm and $\omega(u_p)$ is the vorticity of u_p, defined in (103). Thus we interpret (81) as stating that u_p exhibits a quantized vortex near the origin.

The restriction $\Gamma_t \cong \mathbb{T}^{n-2}$ could be dropped following arguments in [24]. Note, however, that this assumption includes the most important special case, which it when $n = 3$ and Γ_t is a closed string for every t. It should also be straightforward, following [24], to establish analogous results in more general Lorentzian manifolds.

The conclusions of Theorem 3 are much weaker than those of Theorem 1. It only shows that the total energy is divergent, and that the divergent part of the energy concentrates around Γ. One major new difficulty in the vector case is that there are new rotational degrees of freedom, as in the choice of ν^1, ν^2 in (80). This makes it much harder to prove a statement along the lines of "$u_\epsilon \approx U_\epsilon$" for some explicitly constructed U_ϵ.

5.2.   The Abelian Higgs model

The equations of most interest in the cosmological literature on topological defects are gauge theories. Perhaps the simplest example is the Abelian Higgs model, which was the first hyperbolic equation for which it was suggested that solutions should exhibit topological defects. In this discussion we restrict our attention to the physical case of $(1 + 3)$-dimensional Minkowski space.

The Abelian Higgs model concerns a function $\mathfrak{u} : \mathbb{R}^{1+3} \to \mathbb{C}$, called the Higgs field, coupled to a 1-form $\mathcal{A}$ with components $\mathcal{A}_\alpha : \mathbb{R}^{1+3} \to \mathbb{R}$, $\alpha \in 0, \ldots, 3$. We write D_α to denote the covariant derivative

$$D_\alpha\mathfrak{u} := (\partial_\alpha - i\mathcal{A}_\alpha)\mathfrak{u},$$

and we define $\mathcal{F} := d\mathcal{A}$, a 2-form with components

$$\mathcal{F}_{\alpha\beta} := \partial_\alpha\mathcal{A}_\beta - \partial_\beta\mathcal{A}_\alpha.$$

One may regard $\mathcal{A}$ as a $U(1)$ connection and $\mathcal{F}$ as the associated curvature. We write $\langle f, g \rangle$ to denote the *real* inner product

$$\langle f, g \rangle = \text{Re}(f\bar{g}).$$

The Abelian Higgs model is the system

(82) $$-D_\alpha D^\alpha \mathfrak{u} + \frac{\lambda}{4\epsilon^2}(|\mathfrak{u}|^2 - 1)\mathfrak{u} = 0,$$

(83) $$-\epsilon^2 \partial_\alpha \mathcal{F}^{\alpha\beta} - \eta^{\alpha\beta}\langle i\mathfrak{u}, D_\alpha \mathfrak{u} \rangle = 0.$$

where we raise an lower indices with the Minkowski metric, so that

$$D^\alpha \mathfrak{u} := \eta^{\alpha\beta} D_\alpha \mathfrak{u}, \qquad \mathcal{F}^{\alpha\beta} := \eta^{\alpha\gamma}\eta^{\beta\delta}\mathcal{F}_{\gamma\delta}.$$

In (82), (83), the parameter ϵ is a scaling parameter, and plays a very similar role to ϵ in Theorem 1 and 3. On the other hand, the behaviour of solutions depends on the parameter λ in a essential way. In particular, $\lambda = 1$ is a critical case in which the analysis simplifies in some ways.

The following theorem is proved in [18]. In the statement of the theorem we use the notation

$$e_\epsilon(\mathfrak{u}_\epsilon, \mathcal{A}_\epsilon) := \sum_{\alpha=0}^{3} |D_\alpha \mathfrak{u}|^2 + \epsilon^2 \sum_{\alpha,\beta=0}^{3} (F_{\alpha\beta})^2 + \frac{\lambda}{8\epsilon^2}(|\mathfrak{u}|^2 - 1)^2.$$

Theorem 4. *Assume that $T_* < 0 < T^*$, and that $\Gamma \subset (T_*, T^*) \times \mathbb{R}^3$ is a smooth embedded timelike codimension 2 surface of vanishing mean curvature. Assume also Γ_t is homeomorphic to S^1 for every t and that the initial velocity of Γ at $t = 0$ is everywhere 0.*

Let $\lambda > 0$ and $m \in \mathbb{Z}$ be such that conditions (89), (90) below hold, and let $(\mathfrak{u}_\epsilon^{\lambda,m}, \mathcal{A}_\epsilon^{\lambda,m})$ be the associated m-vortex approximate solution around Γ, as described in (92) below

Then given T_-, T_+ such that $T_* < T_- < 0 < T_+ < T^*$, there exists a neighborhood $\mathcal{N}$ of Γ in $(T_-, T_+) \times \mathbb{R}^n$ in which the unsigned distance function δ_Γ is well-defined, and for every $\epsilon \in (0, 1]$, there exists a solution $(\mathfrak{u}_\epsilon, \mathcal{A}_\epsilon)$ of (53) such that

$$\int_{[(T_-,T_+)\times\mathbb{R}^n]\setminus\mathcal{N}} \left[\chi_N \delta_\Gamma^2 + (1 - \chi_{\mathcal{N}})\right] e_\epsilon(\mathfrak{u}_\epsilon, \mathcal{A}_\epsilon)\, dx\, dt \ \leq \ C\epsilon^2,$$

and

$$\int_{[(T_-,T_+)\times\mathbb{R}^n]\setminus\mathcal{N}} |\mathfrak{u}_\epsilon - \mathfrak{u}_\epsilon^{\lambda,m}|^2 + \epsilon^2 |\mathcal{A}_\epsilon - \mathcal{A}_\epsilon^{\lambda,m}|^2\, dx\, dt \leq C\epsilon^2.$$

Moreover,

$$\int_{\{t\}\times\mathbb{R}^n} e_\epsilon(\mathfrak{u}_\epsilon, \mathcal{A}_\epsilon) \geq C$$

for every $t \in \mathbb{R}$.

As with the Theorems 1 and 3 (see Remark 2 above), the proof establishes a number of stability estimates that we have not stated here.

Remark 8. *Hypotheses* (89), (90) *are known to be satisfied for*

- $|m| = 1$ *and* $\lambda \in [\frac{1}{2}, 2]$, *see* [18].
- $|m| = 1$ *and all* λ *larger than some* λ_0, *see* [56].
- $\lambda = 1$ *and all* $m \in \mathbb{Z}$, *see* [32].
- *any* $\lambda > 0$, *and* m *minimizing* $n \mapsto \mathcal{E}_n^\lambda$ *among nonzero integers, see* [18].

They are believed to hold for all $m \in \mathbb{Z}$ when $0 < \lambda < 1$, and for all $\lambda > 0$ when $|m| = 1$.

Conditions (89), (90) on m, λ are related to aspects of the *Euclidean Abelian Higgs model* on $\mathbb{R}^2$, which we now describe. We will use the notation

$$(84) \quad e_{\epsilon,\lambda}^\nu(u, A) := \frac{1}{2}(|D_1 u|^2 + |D_2 u|^2) + \frac{\epsilon^2}{4}(F_{12})^2 + \frac{\lambda}{8\epsilon^2}(|u|^2 - 1)^2$$

for the 2-dimensional Abelian Higgs energy, where $u \in H^1_{loc}(\mathbb{R}^2; \mathbb{C})$ and $A = A_1 dy^1 + A_2 dy^2$ is a 1-form with components in H^1_{loc}, and $F_{12} = \partial_1 A_2 - \partial_2 A_1$. A *finite-energy configuration* is a pair (u, A) such that $e_{\epsilon,\lambda}^\nu(u, A)$ is integrable on $\mathbb{R}^2$.

It is standard to define the vorticity $\omega(u, A)$ to be

$$(85) \quad \omega(u, A) \;=\; \frac{1}{2}\big[\partial_1\langle iu, D_2 u\rangle - \partial_2\langle iu, D_1 u\rangle + F_{12}\big].$$

For every λ, one can check that $|\omega(u, A)| \leq C_\lambda e_{\epsilon,\lambda}^\nu(u, A)$. It is also a fact (see for example [18, Section 2]) that

$$(86) \quad \int_{\mathbb{R}^2} \omega(u, A)\, dy \in \pi\mathbb{Z} \qquad \text{for every finite energy } (u, A).$$

Hence, the space of finite-energy configurations is a disjoint union of sets

$$(87) \quad H_m := \{\text{finite-energy } (u, A) \;:\; \int_{\mathbb{R}^2} \omega(u, A) = \pi m\}$$

called *weak homotopy classes* by Rivière [56] (who gives a different but equivalent description of them). We will use the notation

$$(88) \qquad \mathcal{E}_m^\lambda := \inf\{\int_{\mathbb{R}^2} e_{1,\lambda}^\nu(u, A) \ : \ (u, A) \in H_m\}.$$

The two hypotheses that Theorem 4 imposes on λ, m are

$$(89) \qquad \exists \ (u^{\lambda,m}, A^{\lambda,m}) \in H_m \text{ such that } \int_{\mathbb{R}^2} e_{1,\lambda}^\nu(u^{\lambda,m}, A^{\lambda,m}) = \mathcal{E}_m^\lambda$$

and

$$(90) \qquad \mathcal{E}_m^\lambda \le \mathcal{E}_n^\lambda \quad \text{for all } n \text{ such that } |n| \ge |m|.$$

Assuming that (89) holds, we will define the m-vortex approximate solution to have the form (near Γ)

$$(91) \qquad \mathfrak{u}_\epsilon^{\lambda,m} := u^{\lambda,m}(\frac{\vec{\delta}_\Gamma}{\epsilon}), \qquad \mathcal{A}_\epsilon^{\lambda,m}(y) := \frac{1}{\epsilon}A_a^{\lambda,m}(\frac{\vec{\delta}_\Gamma}{\epsilon})d\delta_\Gamma^a$$

where $\vec{\delta}_\Gamma = (\delta_\Gamma^1, \delta_\Gamma^2)$ is a sort of vector-valued distance function. Equivalently, parallel to (34), the approximate solution admits a simple description in terms of suitable normal coordinates. That is, there exists a neighborhood $\mathcal{N}$ of Γ in $\mathbb{R}^{1+3}$ and a diffeomorphism

$$\phi : (T_-, T_+) \times S^1 \times \{y^\nu \in \mathbb{R}^2 : |y^\nu| < r\} \to \mathcal{N},$$

described in (95), (97) below, such that

$$(92) \qquad \mathfrak{u}_\epsilon^{\lambda,m} \circ \phi(y) = u^{\lambda,m}(\frac{y^\nu}{\epsilon}), \qquad \phi^* \mathcal{A}_\epsilon^{\lambda,m}(y) := \frac{1}{\epsilon}A_i^{\lambda,m}(\frac{y^\nu}{\epsilon})dy^{\nu,i}$$

where $y = (y^0, \dots y^3)$ and $y^\nu := (y^{\nu,1}, y^{\nu,2}) := (y^2, y^3)$. Here $\phi^* A$ denotes the pullback of A by ϕ. (In fact the vector-valued distance function $\vec{\delta}$ in (91) just consists of the last two components of ϕ^{-1}.)

The construction of the approximate solution is not completely canonical, since there is a certain amount of arbitrariness in the construction (95), (97) of the normal coordinate system, or equivalently the function $\vec{\delta}_\Gamma$. However, any two approximate solutions are close enough together that the difference between them scales like a lower-order term, compared to the estimates in Theorem 4.

We remark that approximate solutions of the above form are predicted by "reduced Lagrangian" arguments of the type described in Section 2.7.

The conjecture of Nielsen and Olesen in their landmark 1973 paper [54] was that (82),(83) should have solutions of roughly the form (91), where $(u^{\lambda,m}, A^{\lambda,m})$ is an *equivariant*[3] degree m vortex solution of the Euler-Lagrange equations associated to the 2-d euclidean action functional

$$\int_{\mathbb{R}^2} e^{\nu}_{1,\lambda}(u, A).$$

This is believed to hold for

(93) $0 < \lambda \leq 1$ and $m \in \mathbb{Z}$,   or $\lambda > 1$ and $|m| = 1$.

It follows from Theorem 4 that the Nielsen-Olesen picture holds whenever (89), (90) are satisfied and, in addition, the minimizer $(u^{\lambda,m}, A^{\lambda,m})$ in (89) is equvariant. A well-known conjecture of Jaffe and Taubes [32, Chapter III.1, Conjectures 1 and 2] holds that this holds for all λ, m as in (93). The conjecture is proved when

- $\lambda = 1$ and m is any nonzero integer, see [32].
- λ is larger than some λ_0, and $|m| = 1$, [56].

Thus the exact Nielsen-Olesen scneario is verified for these values of the parameters λ, m.

It is known from work of Berger and Chen [9] that equivariant vortex solutions exist for al $\lambda > 0$ and $m \in \mathbb{Z}$, and a paper of Gustafson and Sigal [27] proves that the equivariant vortex is linearly stable for the full range of parameters (93).

Thus, one might hope to prove that the Nielsen-Olesen scenario holds for the full range (93) of parameters either by

- proving the Jaffe-Taubes conjecture in its entirety, or
- finding a way to prove something like Theorem 4 using the weaker (but still highly nontrivial) stability properties of equivariant solutions from [27] in place of the global minimality (82), and the associated very strong stability properties, used in [18].

§6.   elements of proofs: string/vortex dynamics

There are obvious parallels between the statements of the Theorems 1, 3, and 4 above. Not surprisingly, the proofs have many similarities as well. Indeed, all of them follow more or less the same general template: a change of variables, built around a submanifold Γ solving the relevant

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[3]An equivariant configuration is one that can be written in the form $(\rho(r)e^{im\theta}, a(r)d\theta)$.

geometric evolution problem, that eventually reduces the question of dynamics of codimension k defects in nonlinear field theories to the study of certain stability properties of ground states of associated variational problems in $\mathbb{R}^k$.

This reduction to a problem about stability is very robust. The stability analysis, however, depends intimately on the equation being considered.

Throughout this section we will for concreteness consider $(1 + 1)$-dimensional strings in $\mathbb{R}^{1+3}$.

6.1. the Ginzburg-Landau wave equation

We first consider

$$(94) \qquad u_{tt} - \Delta u + \frac{1}{\epsilon^2} f(|u|^2) u = 0, \qquad u : \mathbb{R}^{1+n} \to \mathbb{R}^2$$

for f as described in Section 5.1.

Fix a codimension 2 timelike extremal submanifold Γ as in Theorem 3, and assume Γ is parametrized by a map $\Psi : (-T_*, T^*) \times S^1 \to \mathbb{R}^{1+3}$ of the form

$$\Psi(y^0, y^1) = (y^0, \psi(y^0, y^1)).$$

Fix smooth maps $\nu_i : (-T_*, T^*) \times S^1 \to \mathbb{R}^{1+3}$ for $i = 1, 2$, such that

$$(95) \qquad \eta_{\alpha\beta} \, \nu_i^\alpha \, \nu_j^\beta = \delta_{ij}, \qquad \eta_{\alpha\beta} \, \nu_i^\alpha \, \partial_{y^a} \Psi = 0, \qquad i, j = 1, 2$$

everywhere in $(-T_*, T^*) \times S^1 \to \mathbb{R}^{1+3}$. Thus, $\{\nu_1(y), \nu_2(y)\}$ forms an orthonormal frame for $(T_{\Psi(y)}\Gamma)^\perp$.

Note that the choice of $\{\nu_1, \nu_2\}$ is rather arbitrary. Indeed, if $\alpha : (-T_*, T^*) \times S^1 \to \mathbb{R}$ is any smooth function, then

$$(96) \qquad \widetilde{\nu}_1 := \cos\alpha \, \nu_1 + \sin\alpha \, \nu_2, \qquad \widetilde{\nu}_2 := -\sin\alpha \, \nu_1 + \cos\alpha \, \nu_2,$$

would also satisfy (95).

Then, parallel to the coordinate system introduced for the codimension 1 case in Remark 1 in Section 2.3, we define

$$(97) \qquad \phi(y^0, \ldots, y^3) := \Psi(y^0, y^1) + y^2 \nu_1(y^0, y^1) + y^3 \nu_2(y^0, y^1).$$

We will write $B^\nu(r) := \{(y^2, y^3) : |(y^2, y^3)|^2 < r^2\}$. We may restrict the domain of ϕ to a set of the form $(-T_1, T^1) \times S^1 \times B^\nu(\rho)$, so that it becomes a diffeomorphism onto its image. Then given a solution u of (94), if we define $v := u \circ \phi$, we find that v solves an equation of the form

$$(98) \qquad -\partial_{y^\beta}(g^{\alpha\beta} \partial_{y^\alpha} v) + b^\alpha \partial_{y^\alpha} v + \frac{1}{\epsilon^2} f(v) = 0,$$

where

(99) $g_{\alpha\beta} := \eta_{\mu\nu}\,\partial_{y^\alpha}\phi^\mu\,\partial_{y^\beta}\phi^\nu, \qquad (g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}, \qquad g = \det(g^{\alpha\beta}).$

and

(100) $\qquad b^\nu(y) := \sqrt{(b^2)^2 + (b^3)^3} \leq C|y^\nu| := C\sqrt{(y^2)^2 + (y^3)^2}.$

This last condition is exactly the fact that Γ is an extremal surface. We define $(a^{\alpha\beta})$ see (62), and

$$e_\epsilon(v) := \frac{1}{|\log\epsilon|}\left[\frac{1}{2}a^{\alpha\beta}\partial_{y^\alpha}v\cdot\partial_{y^\beta}v + \frac{1}{\epsilon^2}F(v)\right].$$

The normalization is chosen so that $\int e_\epsilon(v) = O(1)$ in the regime we are interested in (that is, for functions with a vortex filament.) Then for any smooth solution v of (98), it follows from (100) that
(101)

$$\frac{\partial}{\partial y^0}e_\epsilon(v) \leq \frac{1}{|\log\epsilon|}\left[\sum_{i=1}^{3}\partial_{y^i}\left(g^{i\alpha}\partial_{y^\alpha}v\cdot\partial_{y^0}v\right) + C\left(|D_\tau v|^2 + (y^\nu)^2|D_\nu v|^2\right)\right],$$

where $|D_\tau v|^2 := \sum_{a=0}^{1}(\partial_{y^a})^2$, $|D_\nu v|^2 := \sum_{\alpha=2}^{3}(\partial_{y^\alpha})^2$, and C is uniform on compact subsets of $\mathrm{Domain}(\phi)$. So far, all of this is *exactly* as in the scalar case, apart from the different normalizations.

We are interested in initial data of the form

(102) $\qquad v(0, y^1, \ldots, y^3) \approx q(\frac{y^\nu}{\epsilon}), \qquad \partial_{y^0}v(0, y^1, \ldots, y^3) \approx 0,$

where $q : \mathbb{R}^2 \to \mathbb{R}^2$ is a function such that[4] $q(re^{i\theta}) \approx e^{i\theta}$ as $r \to \infty$, where we identify $\mathbb{R}^2$ and $\mathbb{C}$ in the usual way. This is what is meant by a "string" or "vortex filament".

One might hope to choose q such that $y^0 \mapsto v(y^0)$ is nearly constant — such data would possess a strong stability property. This would however be difficult, in part because it would require a very careful analysis to correctly fix the degree of freedom described in (96).

Thus, we instead note that for data of the form (102), energy concentrates near $S^1 \times \{0\}$, in the sense that for example $e_\epsilon(v(0,\cdot)) \rightharpoonup c_0\delta_{(y^\nu=0)}$ weakly as measures. We will aim to use weighted energy estimates to

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[4]In principle one could also consider q such that $q(re^{i\theta}) \approx e^{id\theta}$ as $r \to \infty$ for some integer $d \geq 2$, corresponding to a multiply quantized vortex filament, but such objects are not expected to be stable.

show that this energy concentration near $\{y^\nu = 0\}$ (that is, near Γ) persists at later times. Following the scalar case, we consider

$$\zeta_1(s;\tau) := \int_{\{s+\tau\}\times S^1} \left(\int_{B^\nu(\rho(s))} (1+|y^\nu|^2)e_\epsilon(v)dy^\nu - c_0 \right) dy^1$$

$$\zeta_2(s;\tau) := \frac{1}{|\log \epsilon|} \int_{\{s+\tau\}\times S^1 \times B^\nu(\rho(s))} |D_\tau v|^2 + (y^n)^2 \left(|\nabla_\nu v|^2 + \frac{F(v)}{\epsilon^2} \right) dy^1 dy^\nu.$$

for a suitable $\rho(s)$ and c_0 (in fact $c_0 = \pi$ is a good choice). One may use (101) to show that

$$\frac{d}{ds}\zeta_1(s;\tau) \le C\zeta_2(s;\tau)$$

provided $s \mapsto \rho(s)$ is taken to decrease sufficiently quickly. To go farther, one needs for example to bound ζ_2 by ζ_1. For this, it is necessary to prove lower bounds for the positive part of ζ_1 that are strong enough to neutralize the negative term $-c_0$ and leave a remainder with some coercivity properties.

The informal idea is to construct a quantity $\zeta_3(s;\tau)$ such that if $\zeta_3(s;\tau)$ is small, then for a large set (whose size is controlled by ζ_3) of points $y^1 \in S^1$,

$$v(s+\tau, y^1, \cdot) \text{ "has a vortex rather near the origin in } \mathbb{R}^{2}\text{",}$$

so that a careful accounting of the energy associated to these vortices may yield the needed lower bound. This is necessary for our arguments, and also yields the qualitative information about u discussed in Remark 7.

The above strategy is similar to the scalar case (compare the discussion of Lemma 8) but the implementation is very different, and relies on machinery for the study of energy and vorticity in (2-dimensional Euclidean) Ginzburg-Landau functionals, developed in [33, 60, 40] and other references. Briefly, given $u \in H^1(B^\nu(\rho); \mathbb{C})$, if u is thought of as a qunatum mechanical wave function, then it is standard to interpret

$$j(u) := Im(\bar{u} \, \nabla u)$$

as the associated momentum density or current, and following [40] and others, we define

(103)
$$\omega(u) := \frac{1}{2}\nabla \times j(u)$$

to be the associated vorticity[5]. Results of [40, 41, 37] develop bounds relating the energy

$$E_\epsilon^{2d}(u; B^\nu(\rho)) := \frac{1}{|\log \epsilon|} \int_{B^\nu(\rho)} \frac{|\nabla_\nu u|^2}{2} + \frac{F(u)}{\epsilon^2} \, dy^\nu$$

and the vorticity. For example, from [41] it is shown that if

$$(104) \qquad \|\omega(u) - \pi\delta_0\|_{W^{-1,1}(B^\nu(\rho))} \le \frac{1}{4}\rho,$$

(one may interpret this as asserting that "there exists a vortex rather near the center of $B^\nu(\rho)$") then there exists a constant such that for $\epsilon \in (0, 1]$

$$(105) \qquad E_\epsilon^{2d}(u; B^\nu(\rho)) \ge \pi - \frac{C}{|\log \epsilon|}.$$

Moreover, this lower bound is attained. These estimates encode a sort of stability property of the set of wave functions u such that

$$(106) \quad E_\epsilon^{2d}(u; B^\nu(\rho)) \le \pi + o(1), \qquad \|\omega(u) - \pi\delta_0\|_{W^{-1,1}(B^\nu(\rho))} \ll \rho.$$

Indeed, since $v \mapsto \omega(v)$ is continuous as a map from H^1 into $W^{-1,1}$, it follows from (104), (105) that if (106) holds, then the only H^1 perturbations of u that can substantially decrease the energy are rather large perturbations that entail "moving the vortex away from the origin."

Guided by these heuristics, and relying on estimates along the above lines from [41, 37], one can formulate and prove weighted energy estimates that provide a crucial ingredient in the proof of Theorem 3, parallel to the role played by Lemma 9 in the proof of Theorem 1. These involve constructing a functional ζ_3 of the form

$$\zeta_3(s; \tau) := \int_{S^1} ||| \, \omega(u(y^1, \cdot)) - \pi\delta_0 \, ||| \, dy^1 \Big|_{y^0 = s + \tau}$$

where $||| \cdot |||$ is a *weighted* $W^{-1,1}$ norm, such that

$$\zeta_2(s; \tau) \le C\big(\zeta_1(s; \tau) + \zeta_3(s; \tau) + |\log \epsilon|^{-1}\big),$$

and such that, in addition, changes in ζ_3 can be controlled by ζ_2 and ζ_1. For details, we refer to [35].

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[5]If we write $u = u^1 + iu^2$, then $\omega(u) = \det \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 \\ \partial_1 u^2 & \partial_2 u^2 \end{pmatrix}$, which is the Jacobian determinant of u. For this reason, it is often denoted Ju.

6.2. the Abelian Higgs model

Finally, we discuss very briefly the proof of Theorem 4.

The argument begins like the proof of Theorem 3, modulo some extra technical complications resulting from the presence of a vector potential A. One performs a change of variables, defined by (97), to rewrite the equations (82), (83) near a timelike extremal surface Γ in terms of coordinates $(y^0, \ldots, y^3) \in (-T_1, T^1) \times S^1 \times B^\nu(\rho)$. The fact that Γ is extremal implies that certain advection terms in the transformed equation vanish when $y^\nu = 0$, and this in turn yields a differential energy inequality with error terms of the form

$$C\big(e_{\epsilon,\lambda}^\tau(u, A) + |y^\nu|^2 e_{\epsilon,\lambda}^\nu(u, A)\big)$$

where $y^\nu = (y^2, y^3)$, and we split the energy (which now depends on a parameter λ as well as ϵ) into parts that are "normal" and "tangential" to Γ. Here (u, A) denotes the solution in the new coordinates

A key new point is to find a quantity, say ζ_3, such that

$$\zeta_3(s; \tau) \text{ small} \implies$$

$$\text{``}(u, A)(s + \tau, y^1, \cdot) \text{ has } m \text{ quanta of vorticity near the origin''}$$

$$\text{for most } y^1 \in S^1$$

$$\implies \text{lower energy bounds,}$$

and such that changes in ζ_3 can be estimated sufficiently well. Again, the strategy is similar to the one used for Theorems 1 and 3, but the details are very different and depend intimately on the structure of the Abelian Higgs model. A sample result, which plays a role similar to that of Lemma 8 in the scalar case, is stated below.

Proposition 1. *Suppose that $\lambda > 0$ and $m \in \mathbb{N}$ satisfy (90). Then for every $R > 0$, there exist constants κ_1 and C, depending on R, λ, m, such that if (u, A) is a finite-energy configuration on $B(R) \subset \mathbb{R}^2$, and if*

$$\pi m - \int_{B(R)} \left(1 - (\frac{|y|}{R})^3\right) \omega(u, A)(y)\, dy < \kappa_1,$$

then

$$\int_{B_\nu(R)} e_{\epsilon,\lambda}^\nu(u, A) \geq \mathcal{E}_m^\lambda - C\epsilon^2$$

for all $\epsilon \in (0, 1]$.

The corresponding results (104), (105) in the ungauged case (94) rely on the large literature on lower bounds for Ginzburg-Landau functionals.

By contrast, lower bounds such as Proposition 1 above are proved more or less by hand in [18].

The interesting regime for Proposition 1, as for Theorem 4, is λ fixed and $0 < \epsilon \ll 1$; this is the scaling relevant in applications to possible cosmic strings. In the asymptotic regime ϵ fixed, $\lambda \gg 1$, the relationship between energy and vorticity is well-understood, due for example to works of Sandier and Serfaty [61]. This situation is in fact similar in spirit to the ungauged Ginzburg-Landau wave equation discussed in Section 6.1.

§7.   timelike submanifolds of Minkowski space

In this section we discuss some aspects of timelike submanifolds of Minkowski space, and in particular extremal surfaces, defined below. Our goal is twofold: first, we attempt to give some background for results, described elsewhere in this paper, about dynamics of interfaces in semilinear hyperbolic equations. Second, we discuss some recent work of Bellettini, Novaga, and Orlandi [8] that initiates a measure-theoretic framework for the study of extremal surfaces and related problems, and we highlight some of the many open questions in this area.

7.1.   basic definitions and notation

Throughout Section 7 we consider the Minkowski metric $(\eta_{\alpha\beta})$ with $c = 1$ (compare (21)), so that

$$(107) \qquad\qquad (\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \mathrm{diag}(-1, 1, \ldots 1).$$

Apart from this, we will follow notational conventions and definitions, such as those of timelike and spacelike vectors, introduced in Section 2.3,

We are interested in timelike embedded submanifolds $\Gamma \subset \mathbb{R}^{1+n}$ of codimension $k \geq 1$, where Γ is *timelike* if every nontrivial vector normal to Γ is spacelike. (This is consistent with our definition of a timelike hypersurface in In Section 2.3.) We recall that terms such as "normal" and "unit" are always understood with respect to the Minkowski metric, unless explicitly specified otherwise.

As in our arguments above (see for example Section 4.2), it is convenient to parametrize a subset of a timelike submanifold Γ by a map $\Psi : O \to \mathbb{R}^{1+n}$, where O is an open subset of $\mathbb{R}^{1+n_0}$, for $n_0 = n - k$. We may also always assume that ψ is nondegenerate in the sense that $D\Psi(y)$ is an injective linear map at every $y \in O$. It does not entail any

loss of generality[6] to assume that these maps have the form
(108)
$$\Psi(y^0, \ldots, y^{n_0}) = (y^0, \psi(y^0, \ldots, y^{n_0})), \text{ with } \partial_{y^0}\psi \cdot \partial_{y^i}\psi = 0 \text{ for } i \geq 1.$$

Recall that we have defined, for $a, b = 0, \ldots, n_0$,

$$\gamma_{ab} := \eta_{\alpha\beta}\, \partial_{y^a}\Psi^\alpha\, \partial_{y^b}\Psi^\beta \quad (\gamma^{ab}) := (\gamma_{ab})^{-1}, \quad \gamma := \det(\gamma_{ab}).$$

Thus, (γ_{ab}) is the representation in local coordinates of the metric induced on Γ by the Minkowski metric on $\mathbb{R}^{1+n}$. If we assume (108), then

(109) $\quad \gamma_{00} = 1 - (\partial_{y^0}\psi)^2, \qquad \gamma_{0i} = \gamma_{i0} = 0, \qquad \gamma_{ij} = \partial_{y^i}\psi \cdot \partial_{y_j}\psi$

for $i, j \geq 1$, at every $y \in O$. In particular, the nondegeracy of ψ implies that $(\gamma_{ij})_{i,j=1}^{n_0}$ is positive definite, and one can check that $\gamma_{00} < 0$ (and hence $\gamma < 0$) if and only if Γ is timelike.

7.2. the area functional and extremal surfaces

Given a submanifold $\Gamma = \Psi(O)$ for some $\Psi : O \subset \mathbb{R}^{1+n_0} \to \mathbb{R}^{1+n}$, we define the Minkowskian area of Γ to be

(110) $$\mathcal{A}(\Gamma) := A(\Psi; O) = \int_O \sqrt{|\gamma|}.$$

This definition makes sense, since in fact $\mathcal{A}(\Gamma)$ depends only on Γ, and not on the parametrization Ψ. That is, if $F : O' \to O$ is a diffeomprphism, so that $\Psi \circ F : O' \to \mathbb{R}^{1+N}$ is a new parametrization of Γ, then it follows from basic multi-variable calculus that

(111) $$A(\Psi; O) = A(\Psi \circ F; O').$$

Note also that $\mathcal{A}$ is preserved by isometries of Minkowski space. That is, if $G : \mathcal{A} \to \mathcal{A}$ is an isometry of $\mathbb{R}^{1+N}$, then

(112) $$\mathcal{A}(\Gamma) = A(\Psi; O) = A(G \circ \Psi; O) = \mathcal{A}(G(\Gamma)).$$

---

[6]Indeed, for any t and any $q \in \Gamma_t$, there is an open subset V^t of Γ_t containing q, and a diffeomorphism $\psi^t : O^t \subset \mathbb{R}^{n_0} \to V^t$. For every $y' = (y^1, \ldots, y^{n_0}) \in O^t$, one can check that there is a unique curve $p(\cdot; y') : (t - \delta, t + \delta) \to \mathbb{R}^n$ such that

$$p(s; y') \in \Gamma_s \text{ for all } s, \quad p(t; y') = \psi^t(y'), \qquad \partial_t p(t; y') \perp T_{p(t)}\Gamma_t \text{ in } \mathbb{R}^n.$$

This verification requires an argument that uses the assumption that Γ is timelike. Then one can define $\psi(y^0, \ldots y^{n_0}) := p(y^0; y')$ on an open neighborhood $O \subset \mathbb{R}^{1+n_0}$ of $(t, (\psi^t)^{-1}(q))$ on which this definition makes sense, and this map has the stated properties.

It is alsoeasy to see that for any bounded open $\Omega \subset \mathbb{R}^{n_0} \times \{0^k\}$,

$$(113) \qquad \mathcal{A}((a,b) \times \Omega) = (b-a)\,\mathcal{L}^{n_0}(\Omega).$$

Any notion of "area" in Minkowski space should possess the symmetries (111), (112), and it is not hard to see that these completely determine the Minkowskian area, up to a normalization factor that we fix with the (natural) condition (113), for any submanifold that can be approximated sufficiently well by a sequence of piecewise afffine submanifolds.

   The area of a submanifold that is not contained in the image of a single map Ψ can be computed as in the Riemannian case by using the above formula together with a partition of unity.

   We will say that a timelike submanifold that is a critical points of the area functional is an *extremal*[7] *surface*. The equation satisfied by extremal surfaces can be written in local coordinates as

$$(114) \qquad \frac{1}{\sqrt{|\gamma|}}\frac{\partial}{\partial y^a}\left(\sqrt{|\gamma|}\gamma^{ab}\frac{\partial \Psi^\alpha}{\partial y^b}\right) = 0, \qquad \alpha = 0,\ldots,n$$

As we discuss in more detail in Section 7.5 below (see Lemma 12), the left-hand side is exactly the Minkowskian mean curvature of Γ at $\Psi(y)$.

   Considered as a partial differential equation, (114) is underdetermined, since any reparametrization of a solution is still a solution. One normally removes this degeneracy by imposing additional conditions to fix the parametrization. For example, if we insist on (108), then the resulting facts (109) imply that (114) becomes a quasilinear hyperbolic system, in the timelike case $|\partial_{y_0}\psi|^2 < 1$.

   Another way of obtaining a hyperbolic system of equations from (114) is to consider submanifolds Γ that can be written as graphs over $\mathbb{R}^{1+n_0}$. In this case, Ψ has the form

$$\Psi(y) = (y, g(y)), \qquad y \in \mathbb{R}^{1+n_0}, \qquad g : \mathbb{R}^{1+n_0} \to \mathbb{R}^k, \quad k = n - n_0.$$

Much of the literature on extremal surfaces considers this situation. In particular, [13, 45] prove global existence, for small, smooth initial data, of solutions of the equation for extremal graphs in the case $k = 1$ of hypersurfaces. The results of [45] establish global existence for any $n_0 \geq 1$,

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[7]In the literature, such surfaces are sometimes referred to as "minimal", including in some previous works of the author, and sometimes as "maximal". In fact, in general they neither minimize nor maximize any quantity related to the area functional. Also, in the literature, "surface" is sometimes understood to mean a submanifold of dimension $2 = 1 + 1$, whereas here we intend a submanifold of dimension ≥ 2 and codimension ≥ 1.

whereas [13] assumes $n_0 \geq 3$ obtains yields stability and decay estimates under somewhat weaker regularity hypotheses. Global existence for small data for $n_0 \geq 2$ and arbitrary codimension is proved in [2], by arguments similar to those of [45] (but with a little more detail, and hence easier to read for non-experts.)

Quite general results on local existence of smooth immersed submanifolds of vanishing mean curvature in a general globally hyperbolic Lorentzian manifold are proved in [48].

7.3. $n_0 = 1.$

The case $n_0 = 1$ is special, both due to poor decay properties of linear waves in 1 space dimension (this is the reason that some of the above-mentioned results are restricted to $n \geq 2$), and because the extremal surface equation, as first noted by Chang and Mansouri [15] and Mansouri and Nambu [46], is exactly solvable. To explain this, we will mostly restrict our attention to surfaces that admit a nondegenerate parametrization of the form

$$(115) \quad \Psi(y^0, y^1) = (y^0, \psi(y^0, y^1)) \quad \text{for some } \psi : [0, T] \times S_E^1 \to \mathbb{R}^n$$

for some $E > 0$, where $S_E^1 := \mathbb{R}/E\mathbb{Z}$. Most of what we say remain valid however if S_E^1 is replaced by $\mathbb{R}$. It is easy to check that if (115) holds and

$$(116) \quad \partial_{y^0}\psi \cdot \partial_{y^1}\psi = 0, \qquad |\partial_{y^0}\psi|^2 + |\partial_{y^1}\psi|^2 = 1,$$

then $\sqrt{|\gamma|}\gamma^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so as long as γ does not vanish, (114) reduces to the linear wave equation

$$(117) \quad (\partial_{y^0}^2 - \partial_{y^1}^2)\psi = 0.$$

(together with the same equation for $\Psi^0 = y^0$, which holds trivially.)

It follows that if

$$(118) \quad \psi(y^0, y^1) = \frac{1}{2}(a(y^1 + y^0) + b(y^1 - y^0)),$$

where a, b are periodic maps $\mathbb{R} \to \mathbb{R}^n$, of period E say, such that

$$(119) \quad |a'(y^1)| = |b'(y^1)| = 1 \qquad \text{for all } y^1 \in S_E^1$$

then Ψ parametrizes an extremal surface. Indeed, any solution ψ of (117) has the form (118), and a short calculation shows that maps of the form (118) satisfy (116) if and only if (119) holds.

We can use the above observations to give an easy proof of the existence of solutions of the Cauchy problem for (immersed) extremal $(1+1)$-dimensional submanifolds of $\mathbb{R}^{1+n}$ for any n.

Lemma 10. *Assume that $\hat{E} > 0$ and that $(\hat{\psi}_0, \hat{\psi}_1) \in C^2 \times C^1(S^1_{\hat{E}}; \mathbb{R}^n)$ satisfy*

$$\hat{\psi}_0'(y^1) \neq 0, \qquad \hat{\psi}_0'(y^1) \cdot \hat{\psi}_1(y^1) = 0, \qquad |\hat{\psi}_0(y^1)|^2 < 1$$

for all $y^0 \in S^1_{\hat{E}}$.

Then there exists a $T > 0$ and a immersion $\hat{\Psi} : [0, T) \times S^1_{\hat{E}} \to \mathbb{R}^n$ of the form (115) *that solves* (114)*, and such that*

(120) $$\hat{\psi}(0, y^1) = \hat{\psi}_0(y^1), \qquad \partial_{y^0} \hat{\psi}(0, y^1) = \hat{\psi}_1(y^1).$$

This solution is unique in the sense that if $\tilde{\Psi}(y^0, y^1) = (y^0, \tilde{\psi}(y^0, y^1))$ is an extremal immersion and satisfies (120) *on $(0, T) \times S^1_{\hat{E}}$, then $\hat{\Psi}$ is a reparametrization of $\tilde{\Psi}$.*

The condition $\hat{\psi}_0' \neq 0$ is necessary if we want $\hat{\Psi}$ to be an immersion at $t = 0$. As discussed above, we do not sacrifice any generality in assuming $\hat{\psi}_0 \cdot \hat{\psi}_1 = 0$, and having imposed this condition, the assumption that $|\partial_{y^0} \hat{\psi}|^2 < 1$ states that the surface we seek should be timelike at $t = 0$.

The condition that a surface admit a parametrization of the form (115) turns out to be necessary for the uniqueness assertion in Lemma 10. Indeed, in [36] an example is presented of smooth nonunique extremal surfaces with the same initial data (120), in which one of the surfaces has the form (115) locally *but not gobally*. Thus, in the uniquenes result, the topological constraints imposed by assumption (115) are significant.

Our presentation follows [36], which in turn borrows from [6], which as far as we know was the first to prove the uniqueness part of the lemma.

Proof.   First, we define

$$\psi_0 = \hat{\psi}_0 \circ \sigma, \quad \psi_1 = \hat{\psi}_1 \circ \sigma,$$

for σ chosen so that $|\psi_0'|^2 + |\psi_1|^2 = 1$. This equation states that

$$(\sigma')^2 |\psi_0' \circ \sigma|^2 + |\hat{\psi}_1 \circ \sigma|^2 = 1,$$

which is an ODE for σ, and if we impose the additional conditions $\sigma(0) = 0, \sigma' > 0$, then it has a unique solution, and it is easy to see that this solution is a diffeomorphism $S_E^1 \to S_{\hat{E}}^1$ for some[8] $E > 0$.

Having reparametrized the initial data to arrange that the constraints (116) hold at time $t = 0$, if we now write down the solution ψ of the Cauchy problem for wave equation, we find that it respects the constraints at all times, and hence determines an extremal surface. Indeed, ψ has form (118), where a, b satisfy

$$a(y^1) + b(y^1) = \psi_0, \qquad a'(y^1) - b'(y^1) = \psi_1$$

so that

$$a' = \frac{1}{2}(\psi_0' + \psi_1), \qquad b' = \frac{1}{2}(\psi_0' - \psi_1).$$

Then it follows from properties of ψ_0, ψ_1 that (119) holds. As a result $\Psi = (y^0, \psi)$ solves (114) as long as $\gamma \neq 0$. It is also clear from the definitions that Ψ is an immersion when $\gamma \neq 0$, and it is easy to check that if (116) holds, then $\gamma = -|\psi_{y^1}|^4$. Since $\partial_{y^1}\psi_0$ never vanishes and S_E^1 is compact, it follows there exists $T > 0$ such that ψ_{y^1} does not vanish $t < T$.

To find a parametrization of the same surface with the original Cauchy data (120), we simply define $\hat{\psi}(y^0, y^1) = \psi(y^0, \sigma^{-1}(y^1))$.

To prove the uniqueness assertion, first we can use the diffeomorphism σ as above to reduce to the case in which (116) holds when $t = 0$. We may further change variables to impose the condition $\partial_{y^0}\tilde{\psi} \cdot \partial_{y^1}\tilde{\psi} = 0$. Once this is done, one can check that the (nonlinear) Euler-Lagrange equations (114) imply that the condition $|\partial_{y^0}\psi|^2 + |\partial_{y^1}\psi|^2 = 1$ is preserved by the evolution, so that (116) continues to hold as long as $\gamma \neq 0$. We refer to [6] for details. Then (116) and (114) imply that the wave equation (117) is satisfied. This implies the uniqueness assertion. Q.E.D.

Let us say that an *extremal cylinder* is a $(1+1)$-dimensional extremal surface of the form (115). We can extract a great deal of information about extremal cylinders from the explicit formula (118), (119). For example:

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[8]In fact E may be interpreted as the total energy of the string, and is given by

$$E = \int_0^{\hat{E}} (\sigma^{-1})' = \int_0^{\hat{E}} \frac{1}{\sigma' \circ \sigma^{-1}} = \int_0^{\hat{E}} \frac{|\hat{\psi}_0'|}{(1 - |\hat{\psi}_1|^2)^{1/2}}.$$

- There does not exist any globally smooth extremal cylinder in $\mathbb{R}^{1+2}$; any initially smooth cylinder develops discontinuities in the spatial tangent in finite time. See [53].
- In $\mathbb{R}^{1+n}$, $n \geq 4$, for generic initial data (with respect to a natural topology) the corresponding extremal cylinder is globally smoothly immersed. For $n = 3$, roughly speaking, both globally smooth immersed solutions and solutions that develop singularities occur for large sets of initial data (that is, sets with nonempty interior.) See [36].
- A *subrelativistic closed string* is a map of the form (118), where $a, b : S_E^1 \to \mathbb{R}^n$ are maps such that $|a'| \leq 1$ and $|b'| \leq 1$ everywhere. Any subrelativstic closed string is a uniform limit of extremal cylinders. See [14, 6].

These results show *some* smallness hypotheses are necessary for global well-posedness of graph-like extremal surfaces as proved in [2, 13, 45], and that in the more general situation considered in [48], one will not expect global well-posedness.

7.4. lorentzian varifolds

Singular extremal surfaces are expected to arise naturally in certain problems coming from mathematical physics, both due to singularity formation in the equations that govern *immersed* extremal surfaces, as discussed in detail in $1 + 1$-dimensional examples above, and due to possibly singular resolutions of collisions or self-interesctions. Thus it is desirable to have a mathematical framework in which to describe and study singular solutions.

An intriguing proposal in this direction was put forward in recent work of Bellettini, Novaga, and Orlandi [8], who initiate the theory of what they call *Lorentzian*[9] *varifolds*. The aim is to define, by analogy with classical (Euclidean) varifolds, a class of generalized submanifolds that are represented by measures on

$$\mathbb{R}^{1+n} \times T_{1+n_0, 1+n}$$

where $T_{1+n_0, 1+n}$ denotes the set of unoriented *timelike* $(1 + n_0)$ planes in $(1 + n)$-dimensional Minkowski space. One would like the space of such generalized submanifolds to have good compactness properties, and a problem that immediately arises is that $T_{1+n_0, 1+n}$ is not compact in

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[9]In fact, in [8] this theory is only developed in Minkowski space, and certain aspects of the treatment there rely, at least superficially, on fixing standard global coordinates for Minkowski space. But the basic idea can be implemented in more general Lorentzian manifolds.

any reasonable[10] topology. This issue is resolved in [8] by introducing a rather natural compactification of $T_{1+n_0,1+n}$, denoted $\overline{B}_{1+n_0,1+n}$, then defining Lorentzian varifolds to be measures on $\mathbb{R}^{1+n} \times \overline{B}_{1+n_0,1+n}$.

The authors of [8] then go on to define *weakly rectifiable* Lorentzian varifolds, and they obtain a formula in this setting for the first variation of (Minkowskian) area, which in particular yields a definition of *extremal* Lorentzian varifolds. They also find some properties of extremal varifolds, including in particular conservation laws for energy and angular momentum.

Let us say that a *weakly extremal cylinder* is the image $\Gamma = \text{Image}(\Psi)$ of a map Ψ of the form (115), (118), (119), *not necessarily an immersion*. It is proved in [8] that every weakly extremal cylinder can be identified as (the spatial support of) a $(1+1)$-dimensional stationary Lorentzian varifold. One may introduce several possible notions of a singular set of such a cylinder, one of which is

$$Sing := \{\Psi(y^0, y^1) : \text{rank}(\nabla\Psi)(y^0, y^1) < 2\}$$
$$= \{\Psi(y^0, y^1) : \partial_{y^1}\psi(y^0, y^1) = 0\}.$$

(The equality above follows by inspection of the explicit formula for Ψ.) Then Γ is timelike and regularly immersed in an open neighborhood of every point of $\Gamma \setminus Sing$, while at every point of $Sing$, not only does the distinguished parametrization degenerate, but also, one can check, Γ fails to be timelike. A stricter notion of singular set is

$$Sing^* := \{p \in Sing : \lim_{q \in \Gamma, q \to p} \tau(q) \text{ does not exist}\},$$

where $\tau(q) := (\partial_{y^1}\psi/|\partial_{y^1}\psi|) \circ \psi^{-1}$, defined wherever it makes sense. The following conclusions are proved in [36].

- If Γ is a weakly extremal cylinder with initial data (in the sense of (120)) in $C^k \times C^{k-1}$, $k \geq 1$, then

$$\dim(Sing) \leq 1 + \frac{1}{k}$$

  where dim denotes the (Euclidean) Hausdorff dimension.
- This bound is sharp; in fact there exist examples of Γ satisfying the above assumption such that $\dim(Sing) = \dim(Sing^*) = 1 + \frac{1}{k}$.

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[10]One possible topology is obtained by mapping $T_{1+n_0,1+n}$ into the space of $(1+n) \times (1+n)$ matrices corresponding to *Minkowskian* orthogonal projection onto elements of $T_{1+n_0,1+n}$, then using the topology induced from $\mathbb{R}^{(1+n) \times (1+n)}$.

- If Γ is a weakly extremal cylinder in $\mathbb{R}^{1+2}$, then $\dim(Sing^*) \geq 1$.

Note that if a, b in (118) are both C^k, then Ψ is C^k. The upper bound on the dimension of $Sing$ follows from only this, together with a rather general and sharp form of Sard's Theorem which can be found in Federer [23, 3.4.3]. The fact that this bound is sharp is proved by an explicit construction which is inspired by examples used to prove the sharpness of Sard's Theorem, see again [23].

7.5. mean curvature of submanifolds of Minkowski space

We end this section by reviewing some basic definitions and properties of mean curvature of timelike submanifolds of Minkowski space.

Given two vector fields Y, Z on $\mathbb{R}^{1+n}$ with components Y^α and Z^α, we will write[11] $\nabla_Y Z$ to be the vector field with components $Y^\beta \partial_{x^\beta} Z^\alpha$.

Assume that Γ is an embedded timelike $(1 + n_0)$-dimensional submanifold of $\mathbb{R}^{1+n}$. If Y, Z are vector fields defined in a neighborhood of a point $p \in \Gamma$, and Y, Z are both tangent to Γ at points of Γ, then we define

$$\nabla_Y^\Gamma Z = P^\Gamma(\nabla_Y Z), \qquad A(Y, Z) := \nabla_Y Z - \nabla_Y^\Gamma Z,$$

where P^Γ denotes Minkowski orthogonal projection onto $T_p\Gamma$. Thus, for $v \in T_p\mathbb{R}^{1+n}$, we define $P^\Gamma v$ to be the unique vector such that

$$P^\Gamma v \in T_p\Gamma, \qquad \eta_{\alpha\beta} \tau^\alpha (P_v^\Gamma)^\beta = \eta_{\alpha\beta} \tau^\alpha v^\beta \quad \text{ for all } \tau \in T_p\Gamma.$$

In particular, $A(Y, Z)$ is just the orthogonal projection of $\nabla_Y Z$ onto the orthogonal complement of $T_p\Gamma$. The following facts are standard and not hard to check.

- If $Y = \tilde{Y}$ and $Z = \tilde{Z}$ on Γ, then then $A(\tilde{Y}, \tilde{Z}) = A(Y, Z)$.
- If f is a smooth function, then $A(fY, Z) = fA(Y, Z) = A(Y, fZ)$.
- $A(Y, Z) = A(Z, Y)$.

These imply that $A(Y, Z)$ depends only on $Y(p)$ and $Z(p)$, so that $A|_p$ defines a symmetric, bilinear map $T_p\Gamma \times T_p\Gamma \to (T_p\Gamma)^\perp$.

The *mean curvature vector* of Γ at p is defined to be the trace of A at p. Thus, in terms of an arbitrary basis $\{X_0, \ldots, X_{n_0}\}$ for $T_p\Gamma$,

(121) $\qquad \vec{H} :=$ mean curvature vector $= \gamma^{ab} A(X_a, X_b)$,

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[11]Everything in this section remains true for general Riemannian or semi-Riemannian manifolds, the only change being that in general, one defines ∇ to be the Levi-Civita connection associated to the (semi)-Riemannian metric.

where

$$(\gamma^{ab}) := (\gamma_{ab})^{-1} \qquad \gamma_{ab} = \eta_{\alpha\beta} X_a^\alpha X_b^\beta, \qquad a, b = 0, \ldots, n_0.$$

Indeed, it can easily be checked that the right-hand side of (121) is independent of the choice of basis, and it reduces to the familiar expression $\sum_{a=0}^{n_0} A(e_a, e_a)$ with an orthonormal basis $\{e_a\}_{a=0}^{n_0}$.

We next find explicit expressions for the mean curvature at a point $p \in \Gamma$, in the same normal coordinate system used throughout this paper. We recall the construction of the coordinates. Assume that

$$\Psi : O \subset \mathbb{R}^{1+n_0} \to \mathbb{R}^{1+n}$$

is a map that parametrizes an open subset of Γ, with $p \in \text{Image}(\Psi)$. Further assume that Ψ nondegenerate in the sense that its gradient has full rank everywhere. We may fix smooth maps $\nu_i : O \to \mathbb{R}^{1+n_0}$, $i = 1, \ldots k := n - n_0$, such that $\{\nu_i(y)\}_{i=1}^k$ form an orthonormal basis for $(T_{\Psi(y)}\Gamma)^\perp$ at every $y \in O$, exactly as in (95)). Next, we define

$$(122) \qquad \phi(y^0, \ldots, y^n) := \Psi(y^\tau) + \sum_{i=1}^k y^{\nu,i} \, \nu_i(y^\tau)$$

where $y^\tau := (y^0, \ldots, y^{n_0})$ and $y^\nu := (y^{\nu,1}, \ldots, y^{\nu,k}) := (y^{n_0+1}, \ldots, y^n)$. We further define, exactly as in (99),

$$g_{\alpha\beta} := \eta_{\mu\nu} \partial_{y^\alpha} \phi^\mu \partial_{y^\beta} \phi^\nu, \qquad (g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}, \qquad g := \det(g_{\alpha\beta})$$

Let us write $X_\alpha := \frac{\partial \phi}{\partial y^\alpha} \circ \phi^{-1}$ to denote the coordinate vector fields on $\text{Image}(\phi) \subset \mathbb{R}^{1+n}$ associated with $(y^0, \ldots, y^n)$, which we may view as a local coordinate system. It is a special case of a general fact from (semi-) Riemannian geometry that

$$\nabla_{X_\alpha} X_\beta = \Gamma^\mu{}_{\alpha\beta} X_\mu \qquad \text{for } \alpha, \beta \in \{0, \ldots, n\},$$

where $\Gamma^\alpha{}_{\beta\delta}$ denote the Christoffel symbols (which we hope will not be confused with the submanifold Γ)

$$(123) \qquad \Gamma^\alpha{}_{\beta\delta} = \frac{1}{2} g^{\alpha\mu} \Big(\frac{\partial}{\partial y^\beta} g_{\delta\mu} + \frac{\partial}{\partial y^\delta} g_{\mu\beta} - \frac{\partial}{\partial y^\mu} g_{\beta\delta} \Big).$$

By construction, for every $p \in \Gamma$

$$\{X_a\}_{a=0}^{n_0} \text{ is a basis for } T_p\Gamma$$

and
$$\{X_\alpha\}_{\alpha=n_0+1}^n \text{ is a orthonormal basis for } (T_p\Gamma)^\perp.$$

It follows that

(124)
$$g_{ab}(y^\tau, 0) = \gamma_{ab}(y^\tau) \quad \text{if } a, b \le n_0, \qquad g_{\alpha\beta}(y^\tau, 0) = \delta_{\alpha\beta} \quad \text{if } \alpha \text{ or } \beta > n_0,$$

and also that
$$A(X_a, X_b) = \sum_{\mu > n_0} \Gamma^\mu{}_{ab} X_\mu.$$

In particular, using (123) and (124), we find that

(125) $$\vec{H} = \sum_{\mu > n_0} g^{ab} \Gamma^\mu{}_{ab} X_\mu = -\frac{1}{2} \sum_{\mu > n_0} (g^{ab} \partial_{y^\mu} g_{ab}) X_\mu$$

where a, b are summed implicitly from 0 to n_0.

Lemma 11. *Assume that Γ is a timelike hypersurface in $\mathbb{R}^{1+n}$, and let $\bar{\nu}$ be any smooth unit vector field such that $\eta_{\alpha\beta}\bar{\nu}^\alpha\bar{\nu}^\beta \equiv 1$, and $\bar{\nu}(X) \perp T_X\Gamma$ at every $X \in \Gamma$.*

Then everywhere on Γ,
$$\vec{H} = H\bar{\nu} \qquad \text{for } H := -\text{div } \bar{\nu}$$

Proof. We use the coordinate system, and the coordinate vectors $\{X_\alpha\}$, described above. With this notation, $\bar{\nu} = v^\alpha X_\alpha$, where $v^a = 0$ on Γ for $a \le n_0 = n - 1$, and $v^n = 1$ on Γ. Using a general expression for the divergence with respect to a coordinate system, we compute

$$\text{div } \bar{\nu}(\phi(y)) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial y^\alpha}(\sqrt{|g|} v^\alpha)$$

$$= \frac{1}{\sqrt{|g|}} \left(\frac{1}{2} |g|^{-1/2} |g| g^{\beta\gamma} \frac{\partial}{\partial y^\alpha} g_{\beta\gamma} v^\alpha + |g|^{1/2} \frac{\partial}{\partial y^\alpha} v^\alpha \right)$$

$$= \frac{1}{2} g^{\beta\gamma} \frac{\partial}{\partial y^\alpha} g_{\beta\gamma} v^\alpha + \frac{\partial}{\partial y^\alpha} v^\alpha.$$

Since v^n attains its maximum on Γ, clearly $\frac{\partial}{\partial y^n} v^n = 0$ on Γ. And for $a < n$, because v^a vanishes on Γ, so do all tangential derivatives of v^a. Thus, $v^\alpha = \delta_n^\alpha$ on Γ, and repeatedly using (124), we have

$$\text{div } \bar{\nu}(\phi(y)) = \frac{1}{2} g^{\beta\gamma} \frac{\partial}{\partial y^\alpha} g_{\beta\gamma} v^\alpha = \frac{1}{2} g^{\beta\gamma} \frac{\partial}{\partial y^n} g_{\beta\gamma}$$

$$= \frac{1}{2} g^{bc} \frac{\partial}{\partial y^n} g_{bc},$$

where b, c are summed from 0 to n_0. Thus the lemma follows from (125). Q.E.D.

Finally, we prove that a different formula for the mean curvature, given in (114) above, in fact coincides with the definition given in (121).

Lemma 12. *If Γ is a timelike submanifold of $\mathbb{R}^{1+n}$ parametrized by a map $\Psi : O \to \mathbb{R}^{1+n}$, then*

$$\text{(126)} \qquad \vec{H}(\Psi(y)) = \frac{1}{\sqrt{|\gamma|}} \frac{\partial}{\partial y^a} \left(\sqrt{|\gamma|} \gamma^{ab} \frac{\partial \Psi}{\partial y^b} \right),$$

We recall that a, b are summed implicitly from 0 to n_0. The Lemma is also valid in Euclidean space. For submanifolds of a more general (semi-) Riemannian manifold, it remains true as long as $\frac{\partial}{\partial y^\alpha}$ is replaced by a covariant derivative.

Proof. It is convenient to write $\widetilde{H}$ to denote the right-hand side of (126). We define a diffeomorphism ϕ as in (122). We can replace Ψ by ϕ and γ by g in the definition of $\widetilde{H}$, since $\Psi(y^\tau) = \phi(y^\tau, 0)$. Then for any $\alpha \in \{0, \ldots, n\}$, at points of the form $(y^\tau, 0)$ we have

$$\eta_{\mu\nu} \widetilde{H}^\mu \frac{\partial \phi^\nu}{\partial y^\alpha} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial y^a} \left(\sqrt{|g|} g^{ab} \eta_{\mu\nu} \frac{\partial \phi^\mu}{\partial y^b} \frac{\partial \phi^\nu}{\partial y^\alpha} \right)$$
$$- g^{ab} \eta_{\mu\nu} \frac{\partial \phi^\mu}{\partial y^b} \frac{\partial^2 \phi^\nu}{\partial y^a \partial y^\alpha}.$$

Since $g^{ab} \eta_{\mu\nu} \frac{\partial \phi^\mu}{\partial y^b} \frac{\partial \phi^\nu}{\partial y^\alpha} = g^{ab} g_{b\alpha} = \delta^a_\alpha$, we deduce that if $\alpha > n_0$, then

$$\eta_{\mu\nu} \widetilde{H}^\mu \frac{\partial \phi^\nu}{\partial y^\alpha} = -g^{ab} \eta_{\mu\nu} \frac{\partial \phi^\mu}{\partial y^b} \frac{\partial^2 \phi^\nu}{\partial y^a \partial y^\alpha} = -\frac{1}{2} g^{ab} \partial_{y^\alpha} g_{ab}.$$

And if $\alpha \leq n_0$, then

$$\eta_{\mu\nu} \widetilde{H}^\mu \frac{\partial \phi^\nu}{\partial y^\alpha} = \frac{1}{\sqrt{|g|}} \partial_\alpha \sqrt{|g|} - \frac{1}{2} g^{ab} \partial_\alpha g_{ab} = 0$$

by a short computation that uses a standard formula for the derivative of the determinant function.

Then by comparison with our formula (125) for $\vec{H} \circ \Psi$ in normal coordinates, we see that

$$\eta_{\mu\nu} \widetilde{H}^\mu \frac{\partial \phi^\nu}{\partial y^\alpha} = \eta_{\mu\nu} (\vec{H}^\mu \circ \Psi) \frac{\partial \phi^\nu}{\partial y^\alpha}$$

for $\alpha = 0, \ldots, n$, which proves that $\widetilde{H} = \vec{H} \circ \Psi$ as desired. Q.E.D.

§8. Open problems

We close this paper by mentioning some of the many open problems related to the themes discussed above.

8.1. the Abelian Higgs model

A complete resolution of leading-order open problems (more refined problems are mentioned below) about dynamics of vortex filaments in the Abelian Higgs model would follow from a proof of the conjecture of Jaffe and Taubes [32].

Problem 1 (Jaffe-Taubes conjecture). *Prove that for all parameters λ, m satisfying* (93), *the minimization problem* (89) *has a unique soution, and this solution is equivariant.*

So far this is only know for $\lambda = 1$ or $\lambda \geq \lambda_0$, for some large λ_0. As noted above, however, it is proved in [27] that the equivariant vortex is linearly stable whenever (93) holds. This may provide an easier route an analog of Theorem 4 in the entire range for which it is expected to hold. Thus one can ask

Problem 2. *Prove conclusions like those of Theorem 4 for all λ, m satisfying* (93) *using the linear stability result of Gustafson and Sigal* [27] *rather than the Jaffe-Taubes conjecture.*

Either of the two above questions would not just prove an analog of Theorem 4, but would also do so with *equivariant* vortices in the approximate solution (91). The more modest goal of simply extending Theorem 4 (without equvariance) to its full expected range could be accomplished by the following problem.

Problem 3. *Prove that hypotheses* (89), (90) *are satisfied for all λ, m satisfying* (93).

As shown in [18], Problem 3 can be reduced to the study of properties of the function $m \mapsto \mathcal{E}_m^\lambda$, defined in (88). For example, in view of [18, Theorem 4.1], given $\lambda > 0$, to prove that (89), (90) hold for $m = 1$, it suffices to show that $\mathcal{E}_n^\lambda \geq \mathcal{E}_1^\lambda$ for all $n > 1$. A result of this character is proved for a somewhat different problem, one without gauge invariance, by Almog *et al* [3], but the proof does not seem to be easy to adapt to the gauged case.

8.2. Other equations

Defects in a huge range of gauge theories have been explored in the cosmological literature, and for many of these models one expects results parallel to those of Theorem 4 to hold. Proving this, for any specific

model, is likely to require a good understanding aspects of an associated lower-dimensional Euclidean action functional, and in particular of the relationship between energy and "vorticity", in the spirit of Proposition 1 (and related results in [18]) for the Abelian Higgs model. This may involve genuine conceptual issues, the main one being that it is not generally clear, in these models, what the analog of "vorticity" should be. For example, for certain non-Abelian gauge theories, vortices are naturally classified by a degree that is an element of a finite group such as $\mathbb{Z}/n\mathbb{Z}$, rather than an integer, as in the Abelian Higgs model. For a model whose vortices have degrees in $\mathbb{Z}/n\mathbb{Z}$, for example, it is not at all clear whether there is a local notion of vorticity that can play the same role as in the Abelian Higgs case. The simplest case of this scenario arises when $n = 2$, which was discussed in the original paper of Nielsen and Olesen [54].

Problem 4. *Consider the coupled $SU(2)$ Yang-Mills Higgs model in $\mathbb{R}^{1+3}$ discussed in [54, Appendix] (see also [66, Section 4.2.2] for example) and prove the existence of solutions that exhibit $\mathbb{Z}/2\mathbb{Z}$ vortex filaments whose asymptotic dynamics is governed by the Nambu-Goto action.*

Models that give rise to exotic vortices, as discussed above, are still in many cases expected to exhibit strings governed by the Nambu-Goto action. However, a different family of models is expected to give rise to so-called superconducting strings, which in principle carry currents and are coupled to an ambient electromagnetic field. This was first proposed in a very influential paper of Witten [67] and is explained rather clearly in [66]. Basic well-posedness issues for the conjectured asymptotic dynamical law of superconducting strings are still open and are quite unclear.

A completely different class of physical models also give rise to questions about interface dynamics in solutions of scalar hyperbolic equations. Papers investigating dynamics of phase boundaries in materials with memories, propose a variety of equations and present formal derivations of laws of interface motion. Examples include the equation

$$(127) \qquad u_{tt} + d^2 u_t - \Delta u + \epsilon^2 \; \bar{d}^2 \Delta u_t + \frac{1}{\epsilon^2}(u^2 - 1)(u - \epsilon k) = 0$$

(studied in [59]), and a hyperbolic analog of the phase field model (see [58].

Problem 5. *Characterize the dynamics of interfaces in solutions of (127), for well-prepared initial data. Presumably the description given in [59] can be made rigorous.*

As noted in Remark 5, a rigorous analysis of the $d = \bar{d} = 0$ case of (127) is given in [24], and as seen in Section 4, incorporating nonzero d does not present a serious challenge. The phase field models of [58] also give rise to interesting open problems, but these may be hard to analyze.

8.3.   more refined analyses

The scalar wave equation (53) can be seen as an infinite-dimensional Hamiltonian system in the state space $X = H^1 \times L^2(\mathbb{R}^n)$. Arguments based loosely on dynamical systems ideas, such as center manifolds, have been have been spectacularly successful in some other problems (*e.g.* [29, 47, 28]) involving effective dynamics in infinite-dimensional Hamiltonian systems. For example, a widely-used strategy is to split the equation under consideration into a part that is expected to contain the leading-order effective dynamics (essentially, a projection of the equation onto a well-constructed submanifold of the state space X) and an error term that must be estimated. If this program can be carried out for (53), it would in principle yield a more detailed description of interface dynamics than that provided by Theorem 1.

Problem 6. *Carry out an analysis of the scalar wave equation* (53) *or the Abelian Higgs model* (82), (83) *along the lines developed for example in* [28], *as described above.*

In most of the examples [29, 47, 28] cited above, the effective dynamics are governed by a *finite*-dimensional Hamiltonian system, which might describe the positions and velocities, and perhaps finitely many other degrees of freedom, attached to a finite number of objects one can think of as point particles — vortices or solitons. For any of the problems we consider, however, the effective dynamics are formally described by an *infinite-dimensional* Hamiltonian system — the Minkowski extremal surface problem, which admits such a formulation — and this is expected cause difficulties, including perhaps problems with loss of derivatives.

The physics literature also contains a great deal of discussion of issues that are probably beyond the reach, for now, of rigorous mathematical analysis. One such issue is the question of what happens when two strings collide — this is expected generically to result in something called "reconnection", but no rigorous results come close to a description of this, and we will not attempt to formulate even a vague question about it. This question remains open in the parabolic case as well — deep global-in-time results on asymptotics of the Ginzburg-Landau heat flow [11] encompass situations in which vortex filaments collide, but do

not give any concrete information about how these collisions are actually resolved.

8.4. timelike extremal surfaces

The results described in Section 7.4 about timelike extremal surfaces and Lorentzian varifolds suggest a large number of interesting directions for research, some of which may be very challenging.

Given a compact immersed $\Gamma_0 \subset \mathbb{R}^n$ of dimension $n_0 < n$, for $n_0 \geq 1$, and a timelike vector field $v : \Gamma_0 \to \mathbb{R}^{1+n}$ along Γ_0, it is known from [48] that there exists, *locally in time*, a smoothly immersed timelike extremal surface Γ of dimension $1 + n_0$ such that $\Gamma_0 = \{x \in \mathbb{R}^n : (0, x) \in \Gamma\}$, and such that $v(p) \in T_p\Gamma$ for every $p \in \Gamma_0$.

Problem 7. *Consider the Cauchy problem for timelike extremal surfaces with compact initial data, as described above,*

If $n_0 = n - 1$ and $n \geq 3$, does every solution Γ necessarily develop singularities in finite time? This is the case when $n_0 = 1$ and $n = 2$, see [53].

Problem 8. *For the Cauchy problem as described above, does there exist, globally in time, a stationary Lorentzian varifold that can be identified as assuming the Cauchy data at time 0? This is known to be true when $n_0 = 1$ for arbitrary $n \geq 2$, see* [8].

One can also ask about partial regularity of stationary Lorentzian varifolds. The easiest example of this kind of question, given below, is probably already difficult.

Problem 9. *Can one develop any sort of partial regularity theory for stationary rectifiable Lorentzian varifolds of dimension $1 + 1$?*

In particular, to what extent to results proved in [36] *for weakly extremal cylinders remain true for more general $(1 + 1)$-dimensional Lorentzian varifolds?*

Partial regularity results from [36] described above, although suggestive, are of limited scope, and do not suggest any reasonable approach to Problem 9, as they rely entirely on explicit formulas for weakly extremal cylinders. Any more general partial regularity theory is likely to be very subtle, in view for example of the delicate dependence of the size of the singular set on the smoothness of initial data, already evident in the relatively simple cases considered in [36].

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