Quantized vortex filaments in complex scalar fields

Robert L. Jerrard *

Abstract. We survey a family of problems in which one seeks to prove that, for a complex-valued function solving a semilinear partial differential equation, energy concentrates around a codimension 2 submanifold solving a geometric problem. The equations in question arise from physical models, and the energy concentration sets are often naturally interpreted as "quantized vortex filaments." One can hope to describe these vortex filaments in a variety of types of PDE, including elliptic (describing an equilibrium of a physical system), parabolic (often describing flow toward an equilibrium) and hyperbolic or dispersive (describing different kinds of oscillations and wave propagation). There are a large number of results about elliptic and parabolic equations, although some significant open problems remain, and less is known about hyperbolic and (especially) dispersive equations.

Mathematics Subject Classification (2010). Primary 00A05; Secondary 00B10.

Keywords. General mathematics, collection of articles.

1. Introduction

In this note we survey a class of problems in which one seeks to establish a relationship between complex-valued functions solving certain semilinear partial differential equations in a domain of dimension $N \geq 3$, on the one hand, and codimension 2 submanifolds of $\mathbb{R}^N$ that solve certain geometric problems, on the other hand.

The relationship that we have in mind can be informally stated in a number of related ways, including for example:

"most level sets of solutions of the PDE are either nearly trivial or close to a solution of the geometric problem."

or

"energy of solutions of the the PDE concentrates around a submanifold solving the geometric problem."

*The author's research was supported by the National Science and Engineering Research Council of Canada under Operating Grant 261955.

One can attempt to prove this sort of statement for equations of elliptic, parabolic, hyperbolic or Schrödinger type, and one would expect the associated geometric problem to have the same type as the semilinear PDE. Many results of this sort are known for elliptic and parabolic equations, and much less is known about the hyperbolic and Schrödinger cases.

We will focus on these sorts of questions for equations related to the (scaled) Ginzburg-Landau[1] functional

$$E_\varepsilon(u) := \tfrac{1}{|\ln\varepsilon|}\int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(|u|^2-1)^2, \qquad u \in H^1_{loc}(\Omega;\mathbb{C}) \tag{1}$$

where we take Ω to be an open subset of $\mathbb{R}^N$ for some $N \geq 3$; for evolution equations we normally take $\Omega = \mathbb{R}^N$. In this context, it is often natural to interpret an energy concentration set as a "quantized vortex filament". We are interested in this functional, and in the equations below, when $0 < \varepsilon \ll 1$.

Associated to $E_\varepsilon(\cdot)$ we can consider the equations

$$-\Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0 \tag{2}$$

$$u_t - \Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0 \tag{3}$$

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0 \tag{4}$$

$$i|\log\varepsilon|u_t - \Delta u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0. \tag{5}$$

The scaling factor $|\ln\varepsilon|$ in (5) will be motivated later. These are, respectively

- the Euler-Lagrange equation for E_ε;
- the L^2 gradient flow for E_ε;
- the Euler-Lagrange equation for the Minkowskian analog of E_ε, in which the term $|\nabla u|^2$ is replaced $\eta^{\alpha\beta}\,\partial_\alpha u\,\partial_\beta u$, where η denotes the Minkowski metric;
- an infinite-dimensional Hamiltonian system for which the Hamiltonian is exactly the Ginzburg-Landau energy E_e. This equations is known as the Gross-Pitaevskii equation.

For (4) or (5) in high dimensions, it is convenient to replace the nonlinearity $(|u|^2-1)u$ by one of the form $f(|u|^2)u$, where $\mathrm{sign}(f(s)) = \mathrm{sign}(s-1)$, and satisfying growth conditions that render the equation globally well-posed in the energy space.

When one considers the geometry of codimension 2 submanifolds, the formal counterpart of the Ginzburg-Landau functional is simply the functional that associates to a submanifold its $N-2$-dimensional Hausdorff measure:

$$\mathcal{A}(\Gamma) := \mathcal{H}^{N-2}(\Gamma), \qquad \Gamma \text{ a codimension 2 submanifold of } \Omega \subset \mathbb{R}^N. \tag{6}$$

[1] "Ginzburg-Landau" is in some ways a misnomer, but a common and convenient one, and one that we will use frequently.

A precise asymptotic relationship between E_ε and $\mathcal{A}$ (or rather, an extension of $\mathcal{A}$ to a more suitable functional setting) is described in Section 3 below. Associated to this functional, parallel to the above, we can consider the following family of problems for codimension 2 submanifolds of Ω, or of $(0,T)\times\Omega$, in the case of evoution problems:

$$\text{mean curvature } = 0 \tag{7}$$

$$\text{velocity - mean curvature} = 0 \tag{8}$$

$$\text{Minkowskian mean curvature } = 0 \tag{9}$$

$$\text{velocity - } \mathbb{J}(\text{mean curvature}) = 0. \tag{10}$$

These are, at least on a formal level

- the Euler-Lagrange equation for $\mathcal{A}$;
- the L^2 gradient flow for $\mathcal{A}$;
- the Euler-Lagrange equation for the Minkowskian analog of $\mathcal{A}$;
- an infinte-dimensional Hamiltonian system for which the Hamiltonian is exactly the $N-2$ area functional $\mathcal{A}$.

In the Schrödinger-type flow (10), at a point p in an oriented codimension 2 submanifold $\Gamma\subset\mathbb{R}^N$, we write $\mathbb{J}$ for the operator $N_p\Gamma\to N_p\Gamma$ corresponding to rotation by 90 degrees in the 2-dimensional normal plane $N_p\Gamma$ to Γ at p, with the sense of the rotation fixed in some way using the orientation of Γ. In the physical case $N=3$, which is also the only case in which anything nontrivial is known about (10), the equation reduces to the *binormal curvature flow* of curves, which can be written parametrically in the form

$$\partial_t\gamma = \partial_s\gamma\times\partial_{ss}\gamma, \qquad\qquad |\partial_s\gamma|^2 = 1$$

for $\gamma:\mathbb{R}\times M^1\to\mathbb{R}^3$, where M^1 is the circle or the real line. Here $\partial_{ss}\gamma$ is the curvature vector of the curve at the point $\gamma(t,s)$, and $\mathbb{J}=\partial_s\gamma\times$.

As we will recall, results relating the elliptic problems (2) and (7), and the parabolic problems (3) and (8), began to appear 20 or more years ago, and for both classes of problems, deep and definitive results about certain questions have been known for more than 10 years. Nonetheless, some open problems remain even in these classical areas. Another issue that has been well-understood for more than 10 years is the convergence, in a suitable sense, of the family of functionals $E_\varepsilon\to\mathcal{A}$, as $\varepsilon\to 0$. Much less is known about the wave and Schrödinger problems, and the few existing results are mostly rather recent.

We close this introduction by mentioning some of the many related problems that we will not discuss in any detail. These include

- questions about relationships between *real*-valued functions solving equations like (2) - (4), and *hypersurfaces* solving geometric problems like (7) - (9). (In this setting there is no Schrödinger equation, and no notion of binormal). There is a huge body of literature on these problems, which are better-understood than the questions about $\mathbb{C}$-valued functions and codimension 2 submanifolds on which we focus.

- gauge theoretic versions of the above problems. These are significant in many physical applications. In general, for the family of questions that we consider, there are a number of results about $U(1)$ gauge theories, such as the Abelian Higgs model, whereas much less is known about nonabelian gauge theories.
- parallel relations betwen functions $\Omega \subset \mathbb{R}^2 \to \mathbb{C}$ and codimension 2 submanifolds of Ω, *i.e.* collections of points. In this situation, the functional that governs the geometry of the submanifolds, corresponding to $\mathcal{A}$, simply counts the number of points, and the $\varepsilon \to 0$ limit of solutions of equations such as (2) - (5) is governed, at least in certain situations, by the "next-order" energy, which involves interactions between points, see [7].

2. vorticity, energy, and balance laws

2.1. some physical quantities. The equations that we study arise in quantum physics, as models of wave functions associated to superfluids (5) or simplified models (in which the magnetic field is neglected) of superconductors (2), (3). In these settings, the codimension 2 submanifolds in which we are interested are naturally interpreted as "vortex submanifolds".

Indeed, in models coming from quantum mechanics, if $u \in H^1_{loc}(\Omega;\mathbb{C})$ with Ω an open subset of $\mathbb{R}^N$, we introduce the quantities

$$|u|^2 := \text{ density} \tag{11}$$

$$e_\varepsilon(u) := \frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(|u|^2-1)^2 = \text{ energy density} \tag{12}$$

$$ju := -\frac{i}{2}(\bar u\, du - u\, d\bar u) \;=\; \text{momentum 1-form} \tag{13}$$

$$Ju := \frac{1}{2} d\, ju = \text{ vorticity 2-form.} \tag{14}$$

When considering the wave equation (4) we will also encounter the quantity

$$\ell_\varepsilon(u) := \frac{-|\partial_t u|^2 + |\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(|u|^2-1)^2, \tag{15}$$

which is just the Minkowskian analog of the energy density. The names that we have given in (11) - (14) are reasonable on physical grounds. The definition of vorticity has the appealing feature that if we write $u = u^1 + iu^2$, then

$$\begin{aligned} Ju = du^1 \wedge du^2 &= \sum_{i<j} (\partial_i u^1 \partial_j u^2 - \partial_i u^2 \partial_j u^1) dx^1 \wedge dx^j \\ &= \text{ the pullback by } u \text{ of the area form on } \mathbb{C}. \end{aligned} \tag{16}$$

Thus Ju is a sort of Jacobian determinant, which motivates the notation Ju. This has a couple of useful consequences. First, if $N = 3$ and we identify the vorticity vector field as the one dual to Ju in a rather natural sense, then

$$\text{vorticity vector field} \;=\; \nabla u^1 \times \nabla u^2,$$

and it follows that integral curves of the vorticity vector field are exactly level curves of u, where u is smooth enough and nondegenerate. In $\mathbb{R}^N$, it is similarly true that where u is smooth and nondegenerate, one can associate to the vorticity 2-form a distribution of $N-2$-planes, and that level sets of u are integral manifolds of the vorticity distribution. It also follows from (16) that if

$$\varphi \in \mathcal{D}^{N-2}(\mathbb{R}^N) := \{\text{smooth, compactly supported } N-2\text{-forms on } \mathbb{R}^N\ \}$$

then

$$\int_{\mathbb{R}^N} \varphi \wedge Ju \;=\; \int_{z\in\mathbb{C}} \left(\int_{u^{-1}\{z\}} \varphi \right) d\,\text{area} \tag{17}$$

if u is smooth enough, in which case $u^{-1}\{z\}$ is a smooth $N-2$-dimensional submanifold with a natural induced orientation, for almost every $z \in \mathbb{C}$. Formulas of the above sort continue to hold for less smooth u, including $u \in H^1(\Omega;\mathbb{C})$, provided the integrals $\int_{u^{-1}\{z\}} \varphi$ on the right-hand side are understood in a suitable (weak) sense.

2.2. how to picture a vortex filament. The simplest solutions of the elliptic Ginzburg-Landau system (2) that possess a “vortex filament” have the form

$$u_\varepsilon(x_1,\ldots,x_N) = f(\frac{r}{\varepsilon})e^{i\theta}, \qquad \text{where } r = \sqrt{x_1^2 + x_2^2} \text{ and } e^{i\theta} = \frac{x_1 + ix_2}{r} \tag{18}$$

for some smooth f, which among other attributes[2] satisfies

$$f(0) = 0, \qquad f' > 0, \qquad f(s) \to 1 \text{ as } s \to \infty. \tag{19}$$

More generally, formal expansions of the equations (2)-(5) suggest that at least some solutions should, to leading order, resemble a function of the form (18), (19), in a suitable coordinate system adapted to the local geometry of some vortex submanifold, and possibly perturbed by a multiplicative phase or in other ways.

It is straightforward to compute the limiting behaviour, as $\varepsilon \to 0$, of quantities such as the vorticity and energy density for a family of functions of the form (18), (19). One finds that, writing $\Gamma := \{x \in \mathbb{R}^N : x_1 = x_2 = 0\}$,

$$\int \phi \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \;\longrightarrow \pi \int_\Gamma \phi \; d\mathcal{H}^{N-2} \qquad \text{for all } \phi \in C_c(\mathbb{R}^N) \tag{20}$$

$$\int \varphi \wedge Ju_\varepsilon \;\longrightarrow \pi \int_\Gamma \varphi \; d\mathcal{H}^{N-2} \qquad \text{for all } \varphi \in \mathcal{D}^{N-2}(\mathbb{R}^N) \tag{21}$$

$$\int S \;:\; \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{|\log\varepsilon|} \longrightarrow \pi \int_\Gamma S : P^\perp \; d\mathcal{H}^{N-2} \qquad \text{for all } S \in C_c(\mathbb{R}^N; M^{N\times N}) \tag{22}$$

[2] In fact f solves the equation $-f'' - \frac{1}{r}f' + \frac{1}{r^2}f + \frac{1}{\varepsilon^2}(f^2-1)f = 0$, the details of which will not matter for our discussion.

as $\varepsilon \to 0$. In the final assertion above.

- $\nabla u_\varepsilon \otimes \nabla u_\varepsilon$ is the matrix whose i,j entry is $\frac{1}{2}(u_{x_i}\bar u_{x_j} + \bar u_{x_i} u_{x_j})$,
- $A : B$ denotes the inner product $A : B = \mathrm{Tr}(A^T B)$ in the space $M^{N\times N}$ of $N \times N$ matrices,
- at $x \in \Gamma$, $P^\perp(x)$ denotes the matrix corresponding to projection onto the 2-dimensional normal space $(T_x\Gamma)^\perp$.

More generally, the same formal computations mentioned above suggest that (20) - (22) may hold, under good conditions, for sequences of solutions of (2) - (5) (with some modifications for the wave equation (4)), for some Γ that is enough like an $N-2$-dimensional submanifold that one can at least define some version of $T_x\Gamma$ almost everywhere, and hence make sense of $P^\perp$.

2.3. some balance laws. In every case, some parallels between the equation and the geometric problem are visible on the level of balance laws. These can be seen as providing insight into known connections, and lending support to conjectured connections, between the PDEs and the geometric problems.

On the PDE side:

- a solution $u_\varepsilon \in H^1(\Omega;\mathbb{C})$ of the elliptic equation (2) satisfies

$$\int \nabla X : (I - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)}) \ \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \, dx \ = \ 0 \qquad \text{for all } X \in C^\infty_c(\Omega;\mathbb{R}^N). \tag{23}$$

  (This, and the identities below, are written in a way that seeks to emphasize their similarities to the geometric balance laws that follow.)

- a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{C}$ of the parabolic equation (3) satisfies

$$\frac{d}{dt}\int \phi \ \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} dx = -\int \phi \, \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|} \, dx - \int \nabla^2\phi : (I - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)}) \ \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \, dx \tag{24}$$

  for all $\phi \in C^\infty_c(\mathbb{R}^N)$.

- a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{C}$ of the wave equation (4) satisfies

$$\int DX : (\eta - \frac{(\eta\, Du_\varepsilon)\otimes(\eta\, Du_\varepsilon)}{\ell_\varepsilon(u_\varepsilon)}) \ \frac{\ell_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \, dx\, dt \ = \ 0 \tag{25}$$

  for $X \in C^\infty_c(\mathbb{R}^{1+N};\mathbb{R}^{1+N})$, where $Du = (\partial_t u, \nabla u)$ and $\eta = \mathrm{diag}(-1,1,\ldots,1)$ represents the Minkowski metric. This is in fact the exact counterpart of (23) in Minkowski spacetime.

- Finally, a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{C}$ of the Gross-Pitaevskii equation (5) satisfies

$$\frac{d}{dt}\int \varphi \wedge J u_\varepsilon \ = \ \int \nabla(\star d\varphi) : \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{|\log\varepsilon|} \, dx \qquad \text{for all } \varphi \in \mathcal{D}^{N-2}(\mathbb{R}^N) \tag{26}$$

where $\nabla(\star d\varphi)$ denotes the $N \times N$ matrix obtained as the gradient of the vector field $\star d\varphi$ that is dual in a natural way to the $N-1$-form $d\varphi$. In particular, if $N=3$ and we identify the 1-form φ with a vector field, then we may identify $\nabla(\star d\varphi)$ with the 3×3 matrix $\nabla(\nabla\times\varphi)$.

The factor of $|\log\varepsilon|^{-1}$ on the right-hand of (26) side is more or less necessary, for our purposes (compare (22)), and can be seen as the reason for our choice of scaling for the Gross-Pitaevskii equation (5).

On the geometric side, we have a family of balance laws with very similar structure.

- A smooth minimal surface $\Gamma\subset\Omega$ satisfies an identity we can write as

$$\int_\Gamma \nabla X : (I-P^\perp)\, d\mathcal{H}^{N-2} \;=\; 0 \qquad \text{for all } X\in C_c^\infty(\Omega;\mathbb{R}^N). \tag{27}$$

- if $(\Gamma_t)_{t\in(0,\infty)}$ is a smooth family of codimension 2 submanifolds evolving by mean curvature (8), then

$$\frac{d}{dt}\int_{\Gamma_t}\phi\, d\mathcal{H}^{N-2} = -\int_{\Gamma_t}\Big[\phi|H|^2+\nabla^2\phi:(I-P^\perp)\,\Big]d\mathcal{H}^{n-2}dx \tag{28}$$

  for all $\phi\in C_c^\infty(\mathbb{R}^N)$, where H denotes the mean curvature vector along Γ_t.

- If Γ is a smooth timelike submanifold of $\mathbb{R}^{1+N}$ with vanishing mean curvature, then

$$\int_\Gamma DX : (\eta - P^\perp_{mink})\, d\lambda^{1,N-2} \;=\; 0 \qquad \text{for all } X\in C_c^\infty(\mathbb{R}^{1+N};\mathbb{R}^{1+N}) \tag{29}$$

  where $\lambda^{1,N-2}$ denotes the Minkoskian area measure on a codimension 2 submanifold, see (38) below, and $P^\perp_{mink}$ denotes projection with respect to the Minkowski metric onto the (Minkowskian) orthogonal complement of $T_{(t,x)}\Gamma$. Explicitly, the integrand can be written $\partial_\alpha X_\beta(\eta^{\alpha\beta}-n_1^\alpha n_1^\beta - n_2^\alpha n_2^\beta)$, where n_1,n_2 satisfy $\eta_{\alpha\beta}n_i^\alpha n_j^\beta=\delta_{ij}$ and $\eta_{\alpha\beta}n_i^\alpha\tau^\beta=0$ for all $\tau\in T_{(t.x)}\Gamma$. Identity (29) is in fact the exact counterpart of (27) in Minkowski spacetime.

- if $(\Gamma_t)_{t\in(0,\infty)}$ is a smooth enough family of codimension 2 oriented submanifolds evolving by binormal mean curvature (10), then

$$\frac{d}{dt}\int_{\Gamma_t}\varphi = \int_{\Gamma_t}\nabla(\star d\varphi):P^\perp\, d\mathcal{H}^{N-2} \qquad \text{for all } \varphi\in\mathcal{D}^{N-2}(\mathbb{R}^N). \tag{30}$$

2.4. passage to limits on a formal level. Note that if one has a sequence of solutions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of the elliptic equation (2) for which one is somehow able to verify that (20), (22) hold for some limiting Γ, then one can directly deduce from (23) that the limiting Γ satisfies (27).

It is similarly true for the Gross-Pitaevskii equation (5) that the identity (26) should rather directly converge to the identity (30), if one has a sequence of solutions that can *somehow* be shown to satisfy (21) and (22) at every t, for some limiting family $(\Gamma_t)_{t\in(0,T)}$. Of course, one would expect this to be much harder for an equation of Schrödinger type than for an elliptic equation, and indeed this seems to be the case.

For the parabolic equations, to carry out a parallel (or perhaps slightly weaker) passage to the limit, one needs both to know that (20), (22) hold, and to relate the L^2 density $|\partial_t u_\varepsilon|^2$ of the velocity field to the squared mean curvature of the limiting object, and for the hyperbolic equation, one needs Lorenz-invariant analogs of (20), (22). So the parallels between the PDEs and the geometric problems are more subtle for these models, on the level of balance laws, but still plainly visible.

2.5. the necessity of geometric measure theory. In the $\varepsilon \to 0$ limit of equations (2) - (5), the energy/vorticity concentration sets associated to a sequence of solutions are known or believed not to be smooth embedded submanifolds, but rather to have singular points for suitable data. Thus any attempt to describe the global geometry of these concentration sets must employ a notion of solution of the associated geometric problem that is insensitive to singularities. So one is naturally lead to consider weak solutions of the geometric problems (7) - (10).

For submanifolds of codimension at least 2, geometric measure theory provides the most natural[3] framework for studying these weak solutions. Indeed, the balance laws (27) and (28), which appear to have some natural affinity with corresponding PDE identities (23) and (24), form the basis for the definitions of *stationary varifold* and *Brakke flow* — these are geometric measure theory notions of weak solutions of the minimal surface problem and motion by mean curvature, respectively.

In other words, if one is somehow able to carry out the passage to limits discussed in Section 2.4 above, in the elliptic case, the object one ends up with, satisfying (27) or some relaxed version thereof, is *exactly* a stationary varifold. And in the parabolic case, when a parallel passage to limits can be justified, one ends up with an object satisfying (a suitable relaxed version of) the balance law (28), and such an object is *exactly* a Brakke flow. As we describe below, this program has been carried out for both elliptic and parabolic equations, prominently in work of Bethuel and collaborators, such as [8, 12]. (This strategy was first implemented in a pioneering paper of Ilmanen [30] concerning the *scalar* Ginzburg-Landau heat flow.) A key ingredient in papers such as [8, 12] is provided by measure-theoretic analysis of Ambrosio and Soner [3], which shows roughly speaking that the desired conclusions follow in the parabolic (and hence elliptic) cases if one can prove suitable lower density bounds on the limiting energy measure.

In a similar way, the identities (29) and (30) can serve as the basis for definitions of weak solutions of the hyperbolic and Schrdönger type geometric evolution

[3]In the case of *scalar* equations and hypersurfaces, notions of weak solution based on the maximum principle are available at least for elliptic and parabolic problems, and provide an alternative framework for the sort of questions we consider here, see for example [27].

problems (9) and (10). Such weak solutions, motivated in part by the problems that we consider here, have begun only very recently to be developed, see [5] and [35] respectively.

3. convergence of functionals

The following theorem makes precise a sense (known as *Gamma-convergence*) in which the Ginzburg-Landau functionals E_ε converge to the codimension 2 area functional in the limit as $\varepsilon \to 0$. Some definitions from geometric measure theory, used in the statement of the theorem, are collected in Appendix A. For now we simply note that an "*i.m.* rectifiable boundary" may be thought of as a possibly non-smooth, oriented, homologically trivial "submanifold" of $\mathbb{R}^N$, and the mass $\mathbf{M}(\cdot)$ is the natural generalization to this setting of the area functional $\mathcal{A}(\cdot)$ appearing above. In particular, to every smooth, embedded, oriented submanifold $M \subset \mathbb{R}^N$, one uniquely may associate an *i.m.* rectifiable boundary T_M, and then $\mathbf{M}(T_M) = \mathcal{A}(M)$.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 3$.*

1. compactness. *If $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a sequence in $H^1(\Omega;\mathbb{C})$ such that*

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) < \infty$$

then there exists a subsequence ε_k and an $N-2$-dimensional i.m. *rectifiable boundary Γ in Ω, such that*

$$\|Ju_{\varepsilon_k} - \pi\Gamma\|_{C_0^{0,\alpha}(\Omega)^*} \to 0 \qquad \textit{for all } \alpha \in (0,1]. \tag{31}$$

2. lower bound. *If $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ is a sequence in $H^1(\Omega;\mathbb{C})$ such that* (31) *holds for some $N-2$-dimensional* i.m. *rectifiable boundary Γ, then*

$$\liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) \geq \pi\mathbf{M}(\Gamma). \tag{32}$$

3. upper bound. *If Γ is any $N-2$-dimensional* i.m. *rectifiable boundary in Ω, then there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ in $H^1(\Omega;\mathbb{C})$ such that* (31) *holds (without passing to a subsequence) and*

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \pi\mathbf{M}(\Gamma). \tag{33}$$

Moreover, whenever both (31) *and* (33) *hold, one has*

$$\frac{1}{\pi|\log\varepsilon_k|} e_{\varepsilon_k}(u_{\varepsilon_k}) \rightharpoonup \mu_\Gamma \qquad \textit{weakly as measures, as } \varepsilon_k \to 0, \tag{34}$$

where μ_Γ is the mass measure *associated to Γ, see Appendix A.*

The compactness and lower bound assertions were first proved in [37], and the upper bound is due to [1], which in fact proved substantially more general results and also presented new proofs of the other parts of the theorem. A key ingredient in both proofs is provided by "vortex ball" constructions and associated estimates from [31, 52]. Some of the results in these references assume that Ω is bounded, but this restriction is unnecessary and has been dropped in later work.

Remark 3.2. The theorem implies various bounds for the vorticity Ju_ε in terms of the energy $E_\varepsilon(u_\varepsilon)$. For example, arguments in [37] easily imply that there exists some $\alpha, C > 0$ such that for any compact $K \subset \Omega$,

$$\limsup_{\varepsilon\to 0} \|\eta_{\varepsilon^\alpha} * Ju_\varepsilon\|_{L^1(K)} \ \le\ CE_\varepsilon(u_\varepsilon).$$

Here $\eta_{\varepsilon^\alpha}$ is a smoothing kernel supported on a ball of radius ε^α. This should be contrasted with the obvious estimate

$$\|Ju_\varepsilon\|_{L^1(\Omega)} \ \le\ C\int_\Omega |\nabla u_\varepsilon|^2 \ \le\ C|\log\varepsilon| E_\varepsilon(u_\varepsilon),$$

which is sharp in the sense that for any $\varepsilon \in (0,1]$, one can construct u_ε such that $\|Ju_\varepsilon\|_{L^1(\Omega)} \ge c|\log\varepsilon|E_\varepsilon(u_\varepsilon)$.

Thus, the energy (scaled as above) does *not* control the total vorticity $\|Ju_\varepsilon\|_{L^1}$. But by averaging on appropriate small scales, one can exploit cancellations to gain a factor $|\log\varepsilon|^{-1} \ll 1$ (when $\varepsilon \lll 1$), and as a result L^1 norm of the "macroscopic part" of the vorticity $\eta_{\varepsilon^\alpha} * Ju$ is indeed controlled by the energy.

Remark 3.3. The convergence in Theorem 3.1 is weak enough that is does not imply anything about convergence of solutions of any of the evolution equations (3)-(5) to the geometric evolution problems (8)-(10), and has only limited implications about convergence of the elliptic problems, as we discuss below. Nonetheless, estimates from Theorem 3.1 and related results provide an important ingredient in many results about the elliptic or parabolic problems, and in all the (much smaller number of) known results about wave and Schrödinger problems.

4. elliptic problems

One can ask two sorts of questions about the relationship between the elliptic Ginzburg-Landau equation (2) and the geometry of minimal surfaces:

- Given a solution of (2) with $0 < \varepsilon \ll 1$, or a sequence of such solutions, do the energy and/or vorticity concentrate near a surface of vanishing mean curvature?

- Given a surface of vanishing mean curvature, can one find a solution of (2) with $0 < \varepsilon \ll 1$ whose energy and/or vorticity concentrate nearby?

4.1. associating minimal surfaces to solutions of (2). Theorem 3.1 directly implies that for sequences of *energy-minimizing* solutions u_ε of (2) with appropriate boundary data, the energy and vorticity concentrate, in the sense of (31) and (34), around a mass-minimizing current. In particular, statements of this type are proved in [1], when one considers the Dirichlet problem for (2) with boundary data in the natural Sobolev space $H^{1/2}(\partial\Omega;\mathbb{C})$.

However, both the earliest and the strongest results linking the elliptic Ginzburg-Landau equation with the minimal surface problem rely not on purely variational techniques, but instead mostly[4] on PDE arguments, including powerful elliptic regularity results tailored to the Ginzburg-Landau setting, initiated and developed by [50, 43, 45] among others. Mature results in this direction, such as [15, 8, 9, 10], show for example that for any sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of solutions of (2), *not* necessarily energy-minimizing and satisfying *only* uniform bounds on the rescaled energy

$$E_\varepsilon(u_\varepsilon) \le C \tag{35}$$

one can extract a subsequence such that (31) holds, and moreover $e_\varepsilon(u_\varepsilon) \rightharpoonup \mu_*$ weakly as measures, where μ_* is a Radon measure that is concentrated and bounded from below on the support of Γ, and has vanishing (generalized) mean curvature in what is known as the *varifold* sense. This result states exactly that the measure μ_* satisfies a relaxed[5], measure-theoretic version of the identity (27) discussed above.

4.2. associating solutions of (2) to minimal surfaces. Surprisingly little is known about the complementary problem. The simplest nontrivial questions of this sort, in the Ginzburg-Landau context, arise when Ω is a smooth, bounded, simply connected, open subset of $\mathbb{R}^3$. Then for $p, q \in \partial\Omega$ we write

$$\ell(p,q) := \text{ the (open) line segment from } p \text{ to } q, \qquad d(p,q) := |p-q|.$$

We say that $\ell(p,q)$ is a *local minimizer* of the "arclength functional" $\mathcal{A}$ if $\ell(p,q)\subset\Omega$ and the pair (p,q) is a local minimizer of $d : \partial\Omega\times\partial\Omega\to\mathbb{R}$. (Thus, we are considering the arclength functional $\mathcal{A}$ with "natural boundary conditions"), We can similarly define a critical point of $\mathcal{A}$, nondegenerate critical point, isolated local minimizer, and so on. One can then ask, given some critical point of $\mathcal{A}$ (possibly satisfying other hypotheses), can one find a nearby solution of the elliptic Ginzburg-Landau system (2) (also with natural boundary conditions, which we tacitly assume for the duration of this section)?

A satisfactory answer to this question is known *only* when $\ell(p,q)$ is an *isolated local minimizer* of $\mathcal{A}$. In this case, it was proved in [46] that for any $\theta\in\mathbb{N}$, there exist (locally energy-minimizing) solutions $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ whose vorticity and energy concentrate in the sense of (31), (34) about the 1-current Γ of constant multiplicity θ associated to the segment $\ell(p,q)$.

[4] Pure PDE techniques are often supplemented by variational estimates. In particular, a common strategy, first developed in [6], studies the 1-form ju, defined in (13), by a Hodge decomposition, with control over d^*ju coming directly from the PDE, and control over $d\,ju = 2Ju$ coming from Theorem 3.1 or related estimates, which do not use any equation.

[5] For example, the measure can have non-constant multiplicity on Γ, and tangent planes may be understood in a weak sense.

The main ingredients in [46] are Theorem 3.1 above and the general Kohn-Sternberg scheme for relating local minimizers and Gamma-convergence [40], and the proof would extend with no difficulties to yield Ginzburg-Landau local minimizers in other, more general situations where one has an *i.m.* rectifiable boundary Γ that is an isolated local minimizer of the mass functional $\mathbf{M}(\,\cdot\,)$ with respect to suitable boundary conditions.

However, as soon as one considers more general critical points[6] only much weaker results are available. In fact, all that is known about this is that if $\ell(p,q)$ is a nondegenerate critical point of $\mathcal{A}$, then there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ of solutions to the Ginzburg-Landau equations (2) such that

$$\frac{1}{\pi}E_\varepsilon(u_\varepsilon) \to d(p,q). \tag{36}$$

This is consistent with the energy and vorticity concentrating around $\ell(p,q)$ in the sense of (31), (34), but is a much weaker statement — one about convergence of critical *values* rather than convergence of critical *points*. These facts are proved in [39], which establishes a general result relating critical points of a limiting functional with those of a sequence of functionals that converges in the sense of Gamma-convergence, then deduces the results described above from Theorem 3.1.

Some related open problems include:

- The paper [46] mentioned above shows that it can happen that $\theta > 1$ vortex filaments cluster around a segment $\ell(p,q)$. Can one give a more precise description of the way in which this clustering occurs? An interesting conjecture and suggestive computations are given in [20].
- Can one improve the results of [39] to construct solutions of (2) for which vorticity and energy can be proved to concentrate around the segment $\ell(p,q)$?

It is worth noting that questions of this sort are very well-understood when one considers *scalar* semilinear elliptic equations and minimal *hypersurfaces*; see for example [23, 24] and [49, 41, 21, 22] respectively, which establish very strong results by some version of Lyapunov-Schmidt reduction. These techniques seem to be difficult to implement, however, for Ginzburg-Landau equations (for $\mathbb{C}$-valued functions), due in part to poor spectral properties of certain linearized operators and (related) difficulties in controlling the phase of complex-valued functions.

5. parabolic problems

The balance laws (24) and (28) mentioned above play an important role in work on the convergence of the parabolic Ginzburg-Landau system (3) to the codimension 2 mean curvature flow (8).

[6] For example, if p,q are two points in $\partial\Omega$ such that $d(p,q) = \mathrm{diam}(\Omega)$ and $\ell(p,q) \subset \Omega$, then $\ell(p,q)$ is a critical point, but not a local minimizer, of the arclength functional.

In particular, the first results in this direction (see [38, 42]) employ an argument first developed by Soner (see the 1995 lectures [56] for the case of a scalar Ginzburg-Landau heat flow) which deduces strong results rather directly from the balance law (24). The main point is a computation which shows that if $(\Gamma_t)_{t\in[0,T]}$ is a codimension 2 mean curvature flow, smoothly embedded in $\mathbb{R}^N$ for every $t \in [0,T]$, then there exist constants $C, \delta > 0$, depending on the geometry of $(\Gamma_t)_{t\in[0,T]}$ but independent of $\varepsilon \in (0,1]$, and a smooth function η such that

$$\eta(t,x) \begin{cases} = \frac{1}{2}\operatorname{dist}^2(x,\Gamma_t) & \text{if } \operatorname{dist}(x,\Gamma_t) \le \delta, \\ \ge \frac{1}{2}\delta^2 & \text{if not,} \end{cases}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^N} \eta(t,x)\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}dx \;\le\; C\int_{\mathbb{R}^N} \eta(t,x)\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}dx \tag{37}$$

for $0 \le t < T$, for any solution u_ε of the Ginzburg-Landau heat flow (3) on $\mathbb{R}^N$. It then follows by Grönwall's inequality that if the energy $e_\varepsilon(u_\varepsilon)$ is concentrated around Γ_0 at time $t = 0$, in the sense that

$$\int_{\mathbb{R}^N} \eta(0,x)\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \to 0 \qquad \text{as } \varepsilon \to 0$$

then it remains concentrated around Γ_t at time t, at least until the first time at which Γ_t develops singularities or self-intersections.

The proof of (37) relies on the fact that when η is taken as a test function in the balance law (24), certain remarkable cancellations occur. These are a consequence of algebraic properties of η which encode the fact that it is built around a codimension 2 mean curvature flow (8). Related ideas were used earlier by Ilmanen [29] and Ambrosio and Soner [2] in proofs of results that establish compatibility between the Brakke flow and other notions of weak solution of mean curvature flow.

The strongest results on the parabolic equation (3) were established in a landmark paper of Bethuel, Orlandi and Smets [12], following important earlier contributions such as [44, 57] and (particularly) the paper [3] of Ambrosio and Soner mentioned above. Bethuel *et al* prove in [12] powerful regularity results which imply that, given any sequence of solutions of the Ginzburg-Landau heat flow in $\mathbb{R}^N$ for any $N \ge 3$, satisfying only uniform energy bounds (35) on the initial data, one can find a subsequence for which the energy measures converge to a Brakke flow, globally in t. As mentioned above, a Brakke flow is a measure theoretic weak solution of the mean curvature flow, and is characterized by the fact that it satisfies a relaxed version of the balance law (28).

Some significant questions are left open by the work described above. For example,

- Assume that $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a sequence of solutions of the Ginzburg-Landau heat flow in $\mathbb{R}^3$, and that the energy measures converge globally to a Brakke flow that can be identified, for $0 \leq t < T_{sing}$, with a smooth curve that develops a self-intersection at time T_{sing}. The results described above show that the energy concentration set is a weak solution of mean curvature flow for all t, but can one say *which* weak solution emerges (maybe only generically) from a self-intersection?

This seems to be a very difficult question, and should probably first be considered in the simplest possible situation, such as for nearly parallel vortex filaments. A related problem, simpler but still very subtle, involving the collision of point vortices in the $2d$ Ginzburg-Landau heat flow, has been analyzed in depth in works of Bethuel, Orlandi, Smets [11, 13, 14] and Serfaty [54, 55].

Another question is:

- Is it possible to give a purely variational proof of the convergence of the Ginzburg-Landau heat flow to codimension 2 mean curvature flow?

General criteria permitting the "Gamma-convergence of gradient flows" have been developed in an important paper of Sandier and Serfaty [53]. These criteria require variational convergence in a significantly stronger sense than is known from the works summarized in Theorem 3.1 – a sort of "C^1 Gamma-convergence" (see [53]). An improvement of Theorem 3.1 of this sort might also shed some light on questions discussed at the end of Section 4.

6. hyperbolic problems

We recall some aspects of the geometry of submanifolds of $(1+N)$-dimensional Minkowski space, which we write $\mathbb{R}^{1+N}$. Suppose that $\mathcal{U} = (a,b) \times U$, for $U \subset \mathbb{R}^{N-k}$ open, and that $\Psi : \mathcal{U} \to \mathbb{R}^{1+N}$ is a nondegenerate map of the form

$$(t,y) \in \mathcal{U} \mapsto \Psi(t,y) = (t, \psi(t,y)) \in \mathbb{R}^{1+N}$$

parametrizing a piece of a codimension k submanifold Γ. We will write

$$g_{ab} := \eta_{\alpha\beta} \partial_a \Psi^\alpha \partial_b \Psi^\beta, \qquad a,b = 0,\ldots,N-k, \quad \alpha,\beta = 0,\ldots,N$$

(here and below, we identify y_0 and t) for the induced metric on Γ, in local coordinates, and

$$g = \det(g_{ab}), \qquad (g^{ab}) = (g_{ab})^{-1}.$$

A submanifold Γ is said to be *timelike* if the induced metric is everywhere Lorentzian (that is, has signature $(-,+,\ldots,+)$.) This holds if and only if $g < 0$ everywhere. The Minkowskian area of a timelike submanifold is defined by

$$\mathcal{A}_{mink}\big(\Psi(\mathcal{U})\big) := \int_{\mathcal{U}} \sqrt{-g}\; dy_0 \ldots dy_{N-k}$$

for the image of a coordinate patch, and more generally may be defined via a partition of unity. In this setting, we also write $\lambda^{1,N-k}$ to denote the Minkowskian area measure, defined by

$$\int_{\Psi(\mathcal{U})} f \; d\lambda^{1,N-k} := \int_{\mathcal{U}} f(\Psi)\sqrt{-g}\, dy_0 \dots dy_{N-k} \qquad \text{for } f \in C_c(\mathbb{R}^{1+N}). \tag{38}$$

Note that all these definitions are independent of the parametrization.

The *Minkowskian mean curvature vector* H_{mink} is the first variation of the Minkowskian area functional. In local coordinates it may be written as

$$H_{mink} = \frac{1}{\sqrt{|g|}}\partial_a \left(\sqrt{|g|}g^{ab}\partial_b\Psi\right).$$

For a timelike submanifold, the equation $H_{mink} = 0$ is a quasilinear hyperbolic equation, in suitable coordinates.

Speculations about relationships between equations like the Ginzburg-Landau wave equation (4) and Minkowskian minimal surfaces date back to the early '70s in the physics literature, starting with a seminal paper of Nielsen and Olesen [48], and entered the (applied) mathematics literature through the formal asymptotic analysis of Neu [47]. The first rigorous results about this problem were given by Bellettini, Novaga and Orlandi [4], who showed that certain measure-theoretic estimates, if they could be proved, would suffice to justify the passage to limits from (25) to (29), or a relaxed version thereof, suitable for describing singular surfaces of vanishing Minkowskian mean curvature.

The strongest results to date about the dynamics of vortex filaments in the Ginzburg-Landau wave equation are summarized in the following.

Theorem 6.1. *Let $\Gamma \subset (-T,T)\times\mathbb{R}^N$ be a smooth embedded timelike codimension $k = 2$ submanifold such that $\Gamma_t := \{x \in \mathbb{R}^N : (t,x) \in \Gamma\}$ is compact for every t and $H_{mink}(\Gamma) = 0$.*

Then for every $\varepsilon \in (0,1]$, there exists a solution of the Ginzburg-Landay wave equation (4) such that for any $T_0 < T$, there is a constant C, depending on Γ and T_0 but inependent of ε, for which

$$\int_{(-T_0,T_0)\times\mathbb{R}^N} \tilde{d}^2 \left(\frac{|u_t|^2 + |\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2}\right)\, dt\; dx \le C \tag{39}$$

where $\tilde{d}(t,x) = \min\{1, \operatorname{dist}((t,x),\Gamma)\}$, and

$$\left| \int S : (\eta - \frac{(\eta\, Du_\varepsilon)\otimes(\eta\, Du_\varepsilon)}{\ell_\varepsilon(u_\varepsilon)}) \;\; \frac{\ell_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\; dx\, dt \right.$$
$$\left. - \int_\Gamma S : (\eta - P^\perp_{mink})\; d\lambda^{1,N-2} \right| \;\le\; \frac{C}{|\ln\varepsilon|^{1/2}}\|S\|_{W^{1,\infty}} \tag{40}$$

for all $S \in C_c^\infty(\mathbb{R}^{1+N}; M^{(1+N)\times(1+N)})$.

The significance of (40) is that it not only establishes that the wave equation balance law (25) converges to its geometric counterpart (29) as long as the associated submanifold Γ remains smooth, but it also provides an estimate, in certain negative Sobolev norms, of the rate of convergence.

Conclusion (40) implies in particular that

$$\int_{(-T_0,T_0)\times\mathbb{R}^N} \left(\frac{|u_t|^2 + |\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2} \right) \, dt \, dx \geq c|\log \varepsilon|.$$

In light of this, the first conclusion (39) shows that the logarithmically diverging part of the energy concentrates near Γ, where $\tilde{d}$ vanishes.

In the definition of the function $\bar{d}$ appearing in (39), for simplicity we understand $\operatorname{dist}((t,x),\Gamma)$ with respect to the Euclidean metric on $\mathbb{R}^{1+N}$. One could also use a Minkowsian notion of distance; in some way this would be more natural, but it is a little harder to describe and in any case would yield an equivalent estimate.

Theorem 6.1 was proved in [32], which also proved similar results relating the real scalar wave equation and timelike hypersurfaces with vanishing Minkowksian mean curvature. These results were proved under additional topological restrictions on Γ_t; more recent work [28] has shown (among other extensions of results of [32]) how to remove this assumption in the scalar case, and the arguments carry over with no change to the situation described in Theorem 6.1. The proof is carried out by weighted energy estimates in Gaussian normal coordinates adapted to the submanifold Γ, where "normal" is understood with respect to the Minkowski metric. It relies crucially on variational estimates, in the spirit of Theorem 3.1, that establish certain strong stability properties of "vortex filaments". The weighted energy estimates implicitly use a form of the balance law (25).

We mention a few of the many related open questions.

- Is it ever possible to describe the dynamics of vortex filaments in solutions of the Ginzburg-Landau wave equation, or related equations[7], globally in time or beyond the onset of singularities for associated geometric flow?

Any progress on this will almost surely require a more precise description of vortex filaments than is given in [32], possibly involving a symplectic orthogonal decomposition or similar ideas. Such an approach is followed in the only result of this type that we know of, due to Cuccagna [16], who proves that for the real scalar Ginzburg-Landau wave equation in 3 dimensions with initial data a very small, smooth perturbation of a flat interface, solutions scatter to a flat interface. In this situation the associated geometric dynamics are very nearly trivial. Techniques of this general character were used in the first work [25] to show that interfaces in

[7] This question is probably easiest to address not for (4), but instead for some equation for which better spectral estimates are available, such as the real scalar counterpart of (4) or for a gauge theory such as the Abelian Higgs model, for which some results parallel to those of Theorem 6.1 have been established in [18]. It might also be prudent to consider situations in which the associated geometric evolution has good properties. For example, $1+1$-dimensional timelike surfaces of vanishing mean curvature admit a very explicit description, which for suitable data yields nontrivial but very simple dynamics for large $|t|$.

the real scalar analog of (3) evolve by (codimension 1) mean curvature flow, before the onset of singularities.

- Recently, motivated in part by problems considered here, Bellettini, Novaga and Orlandi [5] introduced the notion of *Lorentzian varifolds*, which in particular yields a definition of weak solutions of the equation "$H_{mink} = 0$" based essentially on the balance law (29), and analogous to the classical theory of (Euclidean) varifolds. Basic issues about this, such as weak-strong uniqueness, remain open, and one may also wonder whether there is any prospect for a reasonable (partial) regularity theory.

7. Schrödinger-type problems

Unlike the problems considered above, the validity of the (expected) relationship between the Gross-Pitaevskii equation (5) and Schrödinger type geometric flow (10) is almost completely open, and may be stated as the following

Conjecture 7.1. *Let $(\Gamma_t)_{t\in[0,T]}$ be a smooth compact embedded binormal curvature flow*[8] *in $\mathbb{R}^3$.*

Then there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of solutions of the Gross-Pitaevskii equation (5) *in* 3 *space dimensions such that*

$$\int \varphi \wedge J u_\varepsilon(t) \to \pi \int_{\Gamma_t} \varphi \qquad \text{for all } \varphi \in \mathcal{D}^1(\mathbb{R}^3). \tag{41}$$

and

$$\int \phi \frac{e_\varepsilon(u_\varepsilon(t))}{|\log \varepsilon|} dx \ \to \pi \int_{\Gamma_t} \phi \, d\mathcal{H}^1 \qquad \text{for all } \phi \in C_c(\mathbb{R}^N). \tag{42}$$

One can more generally pose the conjecture in $\mathbb{R}^N$ for $N \geq 3$, once the equation (5) is modified in a way that guarantees global well-posedness in the energy space.

A stronger form of the conjecture posits that for any sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth solutions of (5), conclusions (41),(42) are valid for all $t \in [0, T]$ as long as they are satisfied when $t = 0$.

The main partial results toward the above conjecture (many, but not all, of which are proved in $\mathbb{R}^N$ for general $N \geq 3$) are the following:

- it is known that there are families of translating solutions of (5) that converge in the sense of (41), (42) to certain translating binormal curvature flows, see [9, 17].

[8] Thus Γ_t is the oriented image of $\gamma(t, \cdot)$ for some $\gamma : [0, T] \times \mathbb{R} \to \mathbb{R}^3$ satisfying

$$\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma, \qquad |\partial_s \gamma|^2 = 1,$$

with $\gamma(t, s) = \gamma(t, s + \ell)$ for some $\ell > 0$, and injective on $\{t\} \times [0, \ell)$ for all $t \in [0, T]$.

- The conjecture is known to hold if (41), (42) hold when $t = 0$, *and* if Γ_0 is a circle, see [33].
- The conjecture holds for *equivariant*[9] solutions of the Gross-Pitaevskii equation such that (41), (42) hold at time $t = 0$, and Γ_0 is a finite union of circles. (See [36], which in fact considers a slightly different problem, but the proof applies with very minor modifications to the situation described here.)
- If (41), (42) hold when $t = 0$, then there exists a family of 1-dimensional *i.m.* rectifiable boundaries $(\Lambda_t)_{t>0}$ such that $\mathbf{M}(\Lambda_t) \le \mathbf{M}(\Lambda_0)$ and, after passing to a subsequence if necessary,

$$\int \varphi \wedge Ju_\varepsilon(t) \to \pi \int_{\Lambda_t} \varphi \qquad \text{for all } \varphi \in \mathcal{D}^1(\mathbb{R}^3) \tag{43}$$

  for all $t \ge 0$, see [33].
- Moreover, *if* $\mathbf{M}(\Lambda_t) = \mathbf{M}(\Lambda_0)$ for all $t \in [0, T]$, then $\Lambda_t = \Gamma_t$ for $t \in [0, T]$. (This follows from results of [33] combined with with the weak-strong uniqueness theorem of [35] described below.)

These results employ a strategy based on attempting a rigorous passage to limits from the balance law (26) for the Gross-Pitaevskii equation to the corresponding balance law (30) for the binormal curvature flow. To fully implement this strategy, one needs good estimates of the quantity

$$\int_{\mathbb{R}^3} S : \frac{\nabla u \otimes \nabla u}{|\log \varepsilon|} - \pi \int_\Gamma S : P^\perp \, d\mathcal{H}^{N-2}, \qquad S \in C_c^\infty(\mathbb{R}^N; M^{N\times N}) \tag{44}$$

compare (26) and (30). Refined variational estimates from [33], in the spirit of Theorem 3.1, show that the quantity appearing in (44) can be controlled by some combination of the vorticity and energy, but these estimates, which are essentially sharp, are too weak to allow a straightforward passage from (26) to (30). These unfavorable estimates, which are one major obstacle (among several) to the proof of Conjecture 7.1, are related to the same bad spectral properties of certain linearized operators that make Lyapunov-Schmidt reduction difficult for the elliptic Ginzburg-Landau equations.

In the above framework, it is natural to consider a definition of weak solutions of the binormal curvature flow — that is, the $N = 3$ case of (10) — based on the balance law (30). One such definition was proposed in [33], and a slightly improved notion of weak solution was recently put forward in [35], which establishes some properties of this class of weak solutions, such as existence of solutions for initial data which is merely a closed rectifiable curve, and the existence of solutions

[9] That is, solutions of the form $u(t, x, y, z) = f(t, r, z)\frac{x+iy}{r}$, where $r = \sqrt{x^2 + y^2}$. Even with this symmetry, vorticity concentration around several circles is much harder to analyze than the case of non-equvariant solutions with vorticity concentrating around a single circle, in which certain key technical obstacles can be circumvented.

that exhibit change of topology. Such weak solutions need not be unique, but it is shown in [35] that they enjoy a *weak-strong uniqueness* property — a weak solution that agrees with a smooth solution (say, $L^\infty([0,T];W^{3,\infty}(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3))$ at time $t=0$ continues to do so as long as the smooth solution does not develop self-intersections. This is proved by defining what might be called a *relative entropy* of the weak solution with respect to the smooth solution, and showing that its growth can be controlled; the proof thus shows not only weak-strong uniqueness, but also that a weak solution that is close to a smooth solution at $t=0$ (in a particular sense) remains nearby for some interval of time. This argument has been adapted in [34] to show that if u,v are Schrödinger maps $S^1\to S^2$ and v is smooth (say, $C([0,T];H^3(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3))$ then

$$\inf_{\sigma\in\mathbb{T}^1}\|u-\tau_\sigma v\|_{L^2}\le C(v)\|u^0-v^0\|_{L^2}.$$

This estimate is interesting partly because the equation supercritical in L^2; the critical space is $H^{1/2}$. Also, u is only required to belong to $L^\infty([0,T];H^{1/2}(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3)$ which, in addition to being a critical space, is more or less the weakest space in which the equation can be given a meaning in the sense of distributions.

The most prominent open question in this area, and indeed in the family of problems discussed in this paper, is Conjecture 7.1. But there are also a number of interesting questions related to the binormal mean curvature flow and its weak solutions, among them:

- Results of [35] imply in particular the existence of weak solutions of the binormal curvature flow in 3 dimensions for initial data a regular planar polygon. A very interesting numerical and theoretical study of this problem is carried out in [19], which produces explicit formulas for candidate weak solutions and investigates number-theoretic properties of these curves. Parallel behavior is well-understood for the linear Schrödinger equation on S^1 (see for example [51]), and also for periodic NLS with subcritical nonlinearities (this can be treated as a perturbation of the linear case, see [26]), but remains mysterious in the (supercritical) setting of the binormal curvature flow. A first step could be to see whether the candidate solutions constructed in [19] are in fact weak solutions in the sense of [35].
- As far as we know, nothing is established about well-posedness, or other properties, of the binormal mean curvature flow when $N\ge 4$, although the equation may be seen as the canonical Schrödinger-type counterpart of the (Eulidean and Minkowskian) minimal surface equations and mean curvature flow.

A. Appendix

We recall some terminology from geometric measure theory, used in the statement of Theorem 3.1 and elsewhere.

A *k-current* in an open set $\Omega \subset \mathbb{R}^N$ is a bounded linear functional on the space $\mathcal{D}^k(\Omega)$ of C^∞ compactly supported k-forms in Ω. An oriented k-dimensional embedded submanifold M in Ω can be associated with the current T_M defined by

$$T_M(\varphi) := \int_M \varphi \qquad\qquad \text{for } \varphi \in \mathcal{D}^k(\Omega). \tag{45}$$

The space of k-currents also contains some objects that are a little less regular, as well as many objects that are *much* less regular. We will encounter only the former, an example of which is the current of the form

$$\varphi \in \mathcal{D}^k(\Omega) \mapsto \int_{F(A)} \varphi = \int_A F^\# \varphi \tag{46}$$

where A is a compact subset of $\mathbb{R}^k$ and $F : A \to \mathbb{R}^N$ is only Lipschitz (in general not injective, for example), and $F^\# \varphi$ denotes the pullback of φ by F. More generally, an *integer multiplicity*[10] *rectifiable* k-current is one that can be obtained as a limit (with respect to the mass norm, defined below) of currents of the form (46). In the case $k = 1$, such currents can always be written as

$$\Gamma(\varphi) = \sum_{i=1}^\infty \int_{\gamma_i} \varphi, \qquad\quad \varphi \in \mathcal{D}^1(\Omega) \tag{47}$$

where γ_i is the (oriented) image of an injective Lipschitz map $(0,1) \to \mathbb{R}^N$, and the collection $(\gamma_i)_{i=1}^\infty$ has finite mass in the sense defined below.

Given a function $u \in H^1(\Omega;\mathbb{C})$, we can associate to the vorticity Ju the $N-2$-current defined by

$$\varphi \in \mathcal{D}^{N-2}(\Omega) \mapsto \int_\Omega \varphi \wedge Ju.$$

This has a geometric interpretation given by (17), as a kind of average of currents associated to level sets of u.

The mass in Ω of a k-current T, denoted $\mathbf{M}_\Omega(T)$, is defined by

$$\mathbf{M}_\Omega(T) := \sup\{T(\varphi) : \varphi \in \mathcal{D}^k(\Omega), \max_x |\varphi(x)| \le 1\}$$

where for concreteness we use the Eulidean norm on k-covectors. We will almost always drop the subscript and simply write $\mathbf{M}$. It is a straightforward consequence of the definition that for T_M as in (45),

$$\mathbf{M}(T_M) = \mathcal{H}^k(M).$$

If M is merely immersed, then we can still define T_M as in (45), and in this case $\mathbf{M}(T_M)$ corresponds to the $\mathcal{H}^k$ measure with a weight that counts the algebraic multiplicity. This is in fact the right notion of "k-dimensional area" in Theorem 3.1.

[10] we will generally use the abbreviation *i.m.*

In particular, for an *i.m.* rectifiable 1-dimensional current Γ, the representation on the right-hand side of (47) can be chosen with the property that

$$\mathbf{M}(\Gamma) = \sum_i \mathcal{H}^1(\gamma_i).$$

Whenever a current T has locally finite mass, there exists a Radon measure μ_T such that

$$\mathbf{M}_U(T) = \mu_T(U) \quad \text{for every open } U \subset \Omega.$$

We will refer to this as the *mass measure*.

The boundary of k-current T is a $k-1$-current ∂T defined by

$$\partial T(\varphi) = T(d\varphi).$$

The definition is arranged so that $\partial T_M = T_{\partial M}$ for T_M as in (45) — this is just Stokes' Theorem.

A k-current T in Ω is said to be *a boundary* if there exists some $k+1$-current S such that $T = \partial S$ in Ω.

References

[1] Alberti, G., Baldo, S., Orlandi, G., Variational convergence for functionals of Ginzburg-Landau type, *Indiana Univ. Math. J.* **5**4 (2005), 1411-1472.

[2] Ambrosio, L., Soner, H.M., Level set approach to mean curvature flow in arbitrary codimension. *J. Differential Geom.* **43** (1996), no. 4, 693-737.

[3] Ambrosio, L., Soner, H.M., A measure-theoretic approach to higher codimension mean curvature flows. Dedicated to Ennio De Giorgi. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), no. 1-2, 27-49 (1998).

[4] Bellettini, G., Novaga, M., Orlandi, G., Time-like minimal submanifolds as singular limits of nonlinear wave equations *Phys. D* **239** (2010), no. 6, 335-339.

[5] Bellettini, G., Novaga, M., Orlandi, G., Time, Lorentzian varifolds and applications to relativistic strings, *Indiana Univ. Math. J.* **61** (2012), no. 6, 2251-2310.

[6] Bethuel, F., Bourgain, J., Brezis, H., Orlandi, G., $W^{1,p}$ estimates for solutions to the Ginzburg- Landau equation with boundary data in $H^{1/2}$, *C. R. Acad. Sci. Paris Sr. I Math.* **333** (2001), no. 12, 1069-1076.

[7] Bethuel, F., Brezis, H., Hèlein, F., *Ginzburg-Landau vortices.* Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston, Inc., Boston, MA, 1994.

[8] Bethuel, F., Brezis, H., Orlandi, G., Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions, *J. Funct. Anal.* **186** (2001), no. 2, 432-520.

[9] Bethuel, F., Orlandi, G., Smets, D., Vortex rings for the Gross-Pitaevskii equation. *J. Eur. Math. Soc.* **6** (2004), no. 1, 17-94.

[10] Bethuel, F., Orlandi, G., Smets, D., Improved estimates for the Ginzburg-Landau equation: the elliptic case, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4** (2005), no. 2, 319-355.

[11] Bethuel, F., Orlandi, G., Smets, D., Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics. *Duke Math. J.* **130** (2005), no. 3, 523-614.

[12] Bethuel, F., Orlandi, G., Smets, D., Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature, *Ann. of Math. (2)* **163** (2006), no. 1, 37-163.

[13] Bethuel, F., Orlandi, G., Smets, D., Quantization and motion law for Ginzburg-Landau vortices, *Arch. Ration. Mech. Anal.* **183** (2007), no. 2, 315-370.

[14] Bethuel, F., Orlandi, G., Smets, D., Dynamics of multiple degree Ginzburg-Landau vortices. *Comm. Math. Phys.* **272** (2007), no. 1, 229-261.

[15] Bourgain, J., Brezis, H., Mironescu, P., $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation, *Publ. Math. Inst. Hautes tudes Sci.* **99** 99 (2004), 1-115.

[16] Cuccagna, S., On asymptotic stability in 3D of kinks for the ϕ^4 model, *Trans. Amer. Math. Soc.* **360** (2008), no. 5, 2581–2614.

[17] Chiron, D., Vortex helices for the Gross-Pitaevskii equation *J. Math. Pures Appl.* (9) **84** (2005), no. 11, 1555-1647.

[18] Czubak, M., Jerrard, R.L., Topological defects in the Abelian Higgs model, preprint, 2013.

[19] de la Hoz, F., Vega, L., Vortex Filament Equation for a Regular Polygon, preprint, 2013.

[20] del Pino, M., Kowalczyk, M., Renormalized energy of interacting Ginzburg-Landau vortex filaments, *J. Lond. Math. Soc. (2)* **77** (2008), no. 3, 647-665

[21] del Pino, M., Kowalczyk, M., Wei, J., On De Giorgi's conjecture in dimension $N \geq 9$. *Ann. of Math. (2)* **174** (2011), no. 3, 1485-1569.

[22] del Pino, M., Kowalczyk, M., Wei, J., Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$. *J. Differential Geom.* **93** (2013), no. 1, 67-131.

[23] del Pino, M., Kowalczyk, M., Pacard, F., Wei, J., The Toda system and multiple-end solutions of autonomous planar elliptic problems. *Adv. Math.* **224** (2010), no. 4, 1462-1516.

[24] del Pino, M., Kowalczyk, M., Wei, J., Yang, J., Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature, *Geom. Funct. Anal.* **20** (2010), no. 4, 918-957.

[25] de Mottoni, P., Schatzman, M., Geometrical evolution of developed interfaces, *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1533–1589.

[26] Erdogan, M.B., Tzirakis, N., Talbot effect for the cubic nonlinear Schrdinger equation on the torus, preprint, 2013.

[27] Evans, L. C., Soner H. M., Souganidis, P. E., Phase transitions and generalized motion by mean curvature, *Comm. Pure Appl. Math.* **45** (1992), no. 9, 1097–1123.

[28] Galvão-Sousa, B., Jerrard, R.L., Accelerating fronts in semilinear wave equations, *Rend. del Circ. Mat. di Palermo*, to appear.

[29] Ilmanen, T., Elliptic regularization and partial regularity for motion by mean curvature, *Mem. Amer. Math. Soc.* **108** (1994), no. 520.

[30] Ilmanen, T., Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* **38** (1993), no. 2, 417-461

[31] Jerrard, R. L. Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.* **30** (1999), no. 4, 721-746.

[32] Jerrard, R. L., Defects in semilinear wave equations and timelike minimal surfaces in Minkowski space. *Anal. PDE* **4** (2011), no. 2, 285-340.

[33] Jerrard, R.L.,. Vortex filament dynamics for Gross-Pitaevsky type equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **1** (2002), no. 4, 733-768.

[34] Jerrard, R.L., Smets, D., On Schrödinger maps from T^1 to S^2, *Ann. Sci. Norm. Supér.* **45** (2012), no. 4, 637-680 (2013).

[35] Jerrard, R.L., Smets, D., On the motion of a curve by its binormal curvature, *J. Eur. Math. Soc.*, to appear.

[36] Jerrard, R.L., Smets, D., Vortex dynamics for the two dimensional non homogeneous Gross-Pitaevskii equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* , to appear.

[37] Jerrard, R. L., Soner, H. M., The Jacobian and the Ginzburg-Landau Energy, *Calc. Var. PDE* **14** (2002), 151-191.

[38] Jerrard, R. L., Soner, H. M., Scaling limits and regularity results for a class of Ginzburg-Landau systems. *Ann. Inst. H. Poincar Anal. Non Linéaire* **16** (1999), no. 4, 423-466.

[39] Jerrard, R.L.., Sternberg, P., Critical points via Γ-convergence: general theory and applications. *J. Eur. Math. Soc.* **11** (2009), no. 4, 705-753.

[40] Kohn, R.V., Sternberg, P., Local minimisers and singular perturbations, *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), no. 1-2, 69-84.

[41] Kowalczyk, M., On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions. *Ann. Mat. Pura Appl. (4)* **184** (2005), no. 1, 17-52.

[42] Lin, F.-H., Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds, *Comm. Pure Appl. Math.* **51** (1998), no. 4, 385-441.

[43] Lin, F.-H., Rivière, T., Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents, *J. Eur. Math. Soc.* **1** (1999), no. 3, 237-311.

[44] Lin, F.-H., Rivière, T., A quantization property for moving line vortices. *Comm. Pure Appl. Math.* **54** (2001), no. 7, 826-850.

[45] Lin, F.-H., Rivière, T., A quantization property for static Ginzburg-Landau vortices. *Comm. Pure Appl. Math.* **54** (2001), no. 2, 206-228.

[46] Montero, J.A., Sternberg, P., Ziemer, W.P., Local minimizers with vortices in the Ginzburg-Landau system in three dimensions, *Comm. Pure Appl. Math.* **57** (2004), no. 1, 99-125.

[47] Neu, J. C., Kinks and the minimal surface equation in Minkowski space. *Phys. D* **43** (1990), no. 2-3, 421-434.

[48] Nielsen, H. B., Olesen, P., Vortex-line models for dual strings. *Nuclear Phys., B* **61**:45-61, 1973.

[49] Pacard, F., Ritoré, M., From constant mean curvature hypersurfaces to the gradient theory of phase transitions. *J. Differential Geom.* **64** (2003), no. 3, 359-423.

[50] Rivière, T., Line vortices in the U(1)-Higgs model, *ESAIM Contrle Optim. Calc. Var.* **1** (1995/96), 77-167.

[51] Rodnianski, I., Fractal solutions of the Schrdinger equation, *Contemp. Math.* **255** (2000), 181-187.

[52] Sandier, E., Lower bounds for the energy of unit vector fields and applications, *J. Funct. Anal.* **152** (1998), no. 2, 379-403.

[53] Sandier, E., Serfaty, S., Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math* **57** (2004), no. 12, 1627-1672.

[54] Serfaty, S., Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow. I. Study of the perturbed Ginzburg-Landau equation. *J. Eur. Math. Soc.* **9** (2007), no. 2, 177-217.

[55] Serfaty, S., Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow. II. The dynamics. *J. Eur. Math. Soc.* **9** (2007), no. 3, 383-426.

[56] Soner, H.M., Front propagation, *Boundaries, interfaces, and transitions (Banff, AB, 1995)*, 185-206, CRM Proc. Lecture Notes, 13, Amer. Math. Soc., Providence, RI, 1998.

[57] Wang, C.Y., Limits of solutions to the generalized Ginzburg-Landau functional. *Comm. Partial Differential Equations* **27** (2002), no. 5-6, 877-906.

Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada

E-mail: rjerrard@math.toronto.edu