# SOME REMARKS ON MONGE-AMPÈRE FUNCTIONS

ROBERT L. JERRARD

Abstract. The space of Monge-Ampère functions is a rather large function space with the property that, if $u$ is a Monge-Ampère function, then the determinant of the Hessian can be identified with a well-defined Radon measure, denoted $\mathrm{Det}\, D^2u$. Moreover, the map $u \mapsto \mathrm{Det}\, D^2u$ is continuous in a natural weak topology on the space of Monge-Ampère functions. These properties make Monge-Ampère functions potentially useful for certain applications in the calculus of variations. We attempt to give a reasonably elementary treatment of a portion of the relevant theory.

## 1. Introduction

This note presents some properties of Monge-Ampère functions, a class of functions introduced by J.H.G Fu in [4, 5] and extended by the author in [12], and one that we believe is potentially useful for certain problems in the calculus of variations. This possible applicability arises from the fact that every locally Monge-Ampère function $u$ has the property that the determinant of the Hessian matrix $D^2u$, as well as all other minors of the Hessian, can be represented by Radon measures in the product space $\Omega \times \mathbb{R}^n$. Moreover, the determinant measure and measures associated with other minors are all continuous with respect to a natural notion of weak convergence. It is the largest known function space with these properties. For example, Monge-Ampère functions need not be convex, nor generated in any obvious way by convex functions, so that one cannot define $\mathrm{Det}\, D^2u$ using any kind of monotonicity properties of the gradient map. Also, Monge-Ampère functions need not belong to $W^{2,p}_{loc}$ for any $p \geq 1$, and so are not regular enough, in general, that one can make sense of $\det D^2u$ as a distribution.

The defining attribute of a Monge-Ampère function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is that one can associate with it an integral current that can be thought of as corresponding to the graph in the product space $\Omega \times \mathbb{R}^n$ of the gradient $Du$. The basic property of Monge-Ampère functions is that, when this current exists, it is unique. This was proved by Fu [4], and recently extended in [12].

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*Date*: March 16, 2007.

The author was partially supported by the National Science and Engineering Research Council of Canada under operating Grant 261955.

Measures corresponding to the determinant and other minors of the Hessian are defined in terms of the action of the associated integral current on certain $n$-forms. The uniqueness property of this current mentioned above means that the measures defined in this fashion are in some sense canonical, and implies the good weak continuity properties mentioned above.

Here we try to present some properties of Monge-Ampère functions in a way that makes them reasonably accessible to readers without an intimate familiarity with geometric measure theory. We start by summarizing some basic facts. The definitions inevitably are stated in terms of integral currents, but some consequences, such as the weak continuity results mentioned above, can be understood without reference to this machinery. These results complement those found in recent papers of Iwaniec [11], Jerrard and Jung [13], and Fonseca and Maly [3], which discuss weak continuity properties of $u \mapsto \operatorname{Det} D^2u$ in the different framework of distributional determinants. Incidentally, all these authors were apparently unaware of Fu's earlier work — this was certainly true of the authors of [13] at the time that it was written.

We go on to prove that piecewise linear functions are Monge-Ampère. This is known to experts and perhaps obvious to at least some non-experts, but we think that it is useful to present an elementary proof that records some facts for which we do not know any good references, and that illustrates the abstract theory in a concrete setting.

We next discuss regularity properties of Monge-Ampère functions. Technical conditions in Fu's work forced him to restrict his attention to Monge-Ampère functions that are locally Lipschitz. The author's generalization [12] of Fu's uniqueness theorem removes this restriction and thereby raises the question of what regularity properties are naturally enjoyed by Monge-Ampère functions. Here we do not give any positive results, but we construct examples of Monge-Ampère functions that fail to be $C^{0,\alpha}$ for $\alpha$ sufficiently close to 1. As far as we know this result is new. The construction depends only on results about piecewise constant functions from the previous section and is entirely elementary. A related issue is to understand the regularity properties that follow from control over the $L^1$ norm of $\det D^2u$ alone (or the mass of $\operatorname{Det} D^2u$, when it is only a measure), that is, without assuming anything about other minors of the Hessian. This is well understood when $u$ is convex, and we recall some of the these results, as we believe that the comparison is instructive.

Next we discuss some results that characterize $\operatorname{Det} D^2u$ as a measure in terms of the topology of level sets of the functions $y \mapsto u(y) - \xi \cdot y$, for $\xi \in \mathbb{R}^n$. Rather general results in this direction were proved in a somewhat different context by Fu [6],[7], Zähle [19], Rataj and Zähle [18] among others. A nice feature of these results is that they clarify the sense in which the determinant measure for a Monge-Ampère function generalizes the classical construction for convex functions. We discuss these results, and we work out in detail a simple example. We have again tried to keep our argument as elementary as possible, though perhaps with less success than elsewhere in this paper. Nonetheless, the results of this section indicate ways in which concrete geometric information is encoded in the integral currents that appear in the definition of Monge-Ampère functions.

The final section collects some open problems.

The current associated with a Monge-Ampère function can be seen as a Cartesian current associated with the gradient $Du$, and satisfying an additional Lagrangian condition — this is what is responsible for the uniqueness. Thus a lot of related material can be found in the work of Giaquinta, Modica and Souček [9] on Cartesian currents.

1.1. **some preliminaries.** We often implicitly sum over repeated indices.

We will write points in $\Omega \times \mathbb{R}^n$ in the form $(x, \xi)$. Thus an $n$-form in $\Omega \times \mathbb{R}^n$ can be written as

$$\phi = \sum_{|\alpha|+|\beta|=n} \phi^{\alpha\beta}(x,\xi) dx^\alpha \wedge d\xi^\beta. \tag{1.1}$$

Here $\alpha, \beta$ are multiindices, so that for example $\alpha = (\alpha_1, \ldots, \alpha_j) \in \mathbb{Z}^j$ with $1 \le \alpha_1 < \ldots < \alpha_j \le n$. We write $|\alpha|$ to denote the length of the multiindex, so that $|\alpha| = j$ in the above example. Also, for example $dx^\alpha = dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_j}$.

Given a multiindex $\alpha$ of length $j$, we write $\bar\alpha$ to denote the complementary multiindex, of length $n - j$, such that $(\alpha, \bar\alpha)$ is a permutation of $(1, \ldots, n)$, and we write $\sigma(\alpha, \bar\alpha)$ to denote the sign of this permutation. One can check that if $u$ is a smooth function $u : \Omega \to \mathbb{R}$ and $G_{Du}$ denotes the graph of $Du$, then for $\phi$ as in (1.1),

$$\int_{G_{Du}} \phi = \sum_{|\alpha|+|\beta|=n} \int_\Omega \phi^{\alpha\beta}(x, Du(x)) \sigma(\alpha,\bar\alpha) M^{\bar\alpha\beta}(D^2 u)\, dx \tag{1.2}$$

where, if $A = (a_{ij})_{i,j=1}^n$ is an $n \times n$ matrix,

$$M^{\bar\alpha\beta}(A) := \det A^{\bar\alpha\beta},$$

and $A^{\bar\alpha\beta}$ is the $|\bar\alpha| \times |\beta|$ (square) matrix whose $i,j$ entry is $a_{\bar\alpha_i, \beta_j}$. We use the convention $M^{00}(A) = 1$. This formula is the starting point of our subject. It can be verified by writing say $W(x) = (x, Du(x))$, so that $G_{Du}$ is just the image $W(\Omega)$. Then

$$\int_{G_{Du}} \phi = \int_{W(\Omega)} \phi = \int_\Omega W^\# \phi,$$

and upon expanding the right-hand side, one arrives at (1.2); see [9] section 2.2.1 for example for a detailed exposition.

Although it is not needed in much of what follows, we recall that an $n$-dimensional current in $\Omega \times \mathbb{R}^n$ is by definition a linear functional that acts on compactly supported $n$-forms in $\Omega \times \mathbb{R}^n$, ie on objects of the form (1.1). An example of such a current is the functional $\phi \mapsto \int_{G_{Du}} \phi$, if $u$ is a given sufficiently smooth function. We will only consider currents that have no boundary in $\Omega \times \mathbb{R}^n$. We say that such a current is locally integral it has locally finite mass and can be represented in terms of an integral over what Federer [2] terms a countably $(\mathcal{H}^n, n)$-rectifiable set, equipped with an integer-valued weight function. This is a kind of regularity condition, allowing us to think of such currents as generalized $n$-dimensional submmanifolds.

In general we mostly follow notation from [2]. We reluctantly follow conventions of geometric measure theory and write $\|T\|$ to denote the total variation measure associated with a current $T$ of locally finite mass. Similarly, we write $\|\mu\|$ for the total variation measure associated with a vector-valued Radon measure $\mu$. If $X$ is a Banach space, we will write $\|\cdot\|_X$ to indicate the norm in $X$, always with a subscript to avoid any possible ambiguity.

1.2. **definition of Monge-Ampère functions.** The definition of a Monge-Ampère function $u$ is stated in terms of a locally integral $n$-current $[du]$ that is required to satisfy a number of conditions. Informally, these state that $[du]$ can be thought of as a generalized submanifold corresponding to the graph of the gradient of $u$, and that it has some weak regularity and finiteness properties. Before stating these conditions, we emphasize that a basic example of a locally Monge-Ampère function is a $C^2$ function $u$; then one can verify that $[du](\phi) := \int_{G_{Du}} \phi$ has the required properties. Indeed, this remains true for $u \in W^{2,n}_{loc}(\Omega)$.

The definition we give below is only used in its full generality in Section 4. We do not recall all the notation, but following the definition we attempt to explain a couple of points. Also, the meaning of conditions such as (1.3) can be extracted from the discussion in Section 4, see for example (4.15). In Section 2, we consider piecewise linear functions, and there the conditions of the definition take on a very concrete form, see (2.4), (2.5), (2.6).

If $\Omega$ is an open subset of $\mathbb{R}^n$, then $u \in W^{1,1}_{loc}(\Omega)$ is said to be locally a Monge-Ampère function in $\Omega$ if there exists an $n$-dimensional locally integral current $[du]$ in $\Omega \times \mathbb{R}^n$ such that

$$\partial[du] = 0 \text{ in } \Omega \times \mathbb{R}^n \tag{1.3}$$

$$[du] \text{ is Lagrangian} \tag{1.4}$$

which one can think of as meaning "weakly curl-free"; see below for more details;

$$\mathbf{M}([du] \llcorner (K \times \mathbb{R}^n)) < \infty \text{ whenever } K \subset \Omega \text{ is compact.} \tag{1.5}$$

and

$$[du](\phi dx^1 \wedge \ldots \wedge dx^n) = \int \phi(x, Du(x))\, dx \tag{1.6}$$

for every $\phi \in C_c^\infty(R^n \times \mathbb{R}^n)$.

We say that $u$ is Monge-Ampère if $\mathbf{M}([du]) < \infty$. We write $u \in MA_{loc}(\Omega)$ if $u$ is locally Monge-Ampère in $\Omega$, and $u \in MA(\Omega)$ if $u$ is Monge-Ampère.

Note that (1.6) states that for a "purely horizontal" $n$-forms $\phi dx^1 \wedge \ldots \wedge dx^n$ (with no $d\xi^j$ s), $[du](\phi)$ exactly agree with integration over the graph of $Du$ as displayed in (1.2) (recalling our convention that $M^{00}(D^2u) := 1$).

Observe also that if $u \in W^{2,n}$ then, as remarked earlier, $u$ is Monge-Ampère with $[du](\phi) := \int_{G_{Du}} \phi$ as given in (1.2), and then the definition of $\mathbf{M}(\cdot)$ implies that

$$\begin{aligned}\mathbf{M}([du] \llcorner (K \times \mathbb{R}^n) &= \int_K \Big( \sum_{|\alpha|+|\beta|=n} [M^{\bar{\alpha}\beta}(D^2u(x))]^2 \Big)^{1/2}\, dx \\ &\approx \sum_{|\alpha|+|\beta|=n} \|M^{\bar{\alpha}\beta}(D^2u)\|_{L^1(K)}.\end{aligned} \tag{1.7}$$

Thus control over the mass of $[du]$ assumed in (1.5) provides control over the $L^1$ norms of all minors of $D^2u$. A suitable generalization of this fact remains true for all Monge-Ampère functions.

We now give the definition of the Lagrangian condition appearing in (1.4). In fact we give two equivalent definitions. Before going into the details, we remark that one can check from (1.11) that if $u$ is smooth, then $\phi \mapsto \int G_{Du}(\phi)$ is Lagrangian, and moreover, this holds exactly because $D^2u = (D^2u)^T$. This motivates the interpretation of (1.4) as weakly curl-free.

An integral $n$-current in $\mathbb{R}^n \times \mathbb{R}^n$ is Lagrangian if $\mathcal{H}^n$ almost every approximate tangent $n$-plane $P$ satisfies

$$\langle \omega, \tau_1 \wedge \tau_2 \rangle = 0 \text{ for any two vectors } \tau_1, \tau_2 \text{ tangent to } P \tag{1.8}$$

for $\omega = \sum_i dx^i \wedge d\xi^i$. An $n$-plane $P$ with this property is said to be Lagrangian. It is not hard to check that $P$ is Lagrangian if and only if for every basis $\{\tau_i\}_{i=1}^n$ of the tangent space to $P$,

$$\langle \omega \wedge \eta, \tau_1 \wedge \ldots \wedge \tau_n \rangle = 0 \qquad \text{for every } n-2\text{-covector } \eta. \tag{1.9}$$

This equivalence is based on the identity

$$\langle \omega \wedge \eta, \tau_1 \wedge \ldots, \wedge \tau_n \rangle = \sum_{\alpha \in I(2,n)} \operatorname{sgn}(\alpha, \bar{\alpha}) \langle \omega, \tau_{\alpha_1} \wedge \tau_{\alpha_2} \rangle \langle \eta, \tau_{\bar{\alpha}_1} \wedge \ldots \wedge \tau_{\bar{\alpha}_{n-2}} \rangle \tag{1.10}$$

From (1.9) one can check that an integral $n$-current $T$ is Lagrangian if and only if

$$T(\omega \wedge \phi) = 0 \text{ for every smooth compactly supported } n-2\text{-form } \phi. \tag{1.11}$$

The advantage of this formulation is that it makes it clear that this property is preserved under weak convergence.

1.3. **basic properties of Monge-Ampère functions.** The *raison d'être* of the space of Monge-Ampère functions is provided by the following

**Theorem 1.1.** *(*[4], [12]*) Suppose that $u \in MA_{loc}(\Omega)$. Then there is* exactly *one current in $\Omega \times \mathbb{R}^n$ satisfying* (1.3)*,* (1.4)*,* (1.5)*,* (1.6)

Fu's original definition was actually slightly different from that given above: in place of (1.5) he assumed the stronger condition

$$(K \times \mathbb{R}^n) \cap \operatorname{supp}[du] \text{ is compact whenever } K \subset \Omega \text{ is compact.} \tag{1.12}$$

Thus in his version of Theorem 1.1, this hypothesis appeared in place of (1.5). As mentioned above, functions satisfying (1.12) are necessarily locally Lipschitz, so this condition is genuinely restrictive. The theorem in the form stated above is proved in [12]. Fu's terminology also differed somewhat — he defined Monge-Ampère functions on $\mathbb{R}^n$, and his definition corresponded to what we are here calling a local Moge-Ampere function.

For general $u \in MA_{loc}(\Omega)$, not necessarily smooth, we use the current $[du]$ to *define* all minors of $D^2u$ as measures in the product space $\Omega \times \mathbb{R}^n$. That is, we can define a measure $M^{\bar\alpha\beta}(D^2u)(dx, d\xi)$ by requiring that

$$\int_{\Omega\times\mathbb{R}^n} \phi(x,\xi)\sigma(\alpha,\bar\alpha)M^{\bar\alpha\beta}(D^2u)(dx,d\xi) := [du](\phi\, dx^\alpha \wedge d\xi^\beta) \tag{1.13}$$

for all $\phi \in C_c^\infty(\Omega \times \mathbb{R}^n)$. This definition is motivated by (1.2). Of particular interest is the special case

$$\int_{\Omega\times\mathbb{R}^n} \phi(x,\xi) \operatorname{Det} D^2u(dx,d\xi) := [du](\phi\, d\xi^1 \wedge \ldots \wedge d\xi^n). \tag{1.14}$$

The definitions imply that

$$\sum_{|\alpha|+|\beta|=n} \|M^{\bar\alpha\beta}(D^2u)\|(K \times \mathbb{R}^n) \approx \mathbf{M}([du] \llcorner (K \times \mathbb{R}^n))$$

so that, as in the smooth case (1.7), the mass of measures associated with minors of $D^2u$ is controlled by the mass of $[du]$ (and conversely).

The following is a straightforward consequence of Theorem 1.1, see [4, 9, 12].

**Corollary 1.1.** *If $u_k$ is a sequence of Monge-Ampère functions on a domain $\Omega \subset \mathbb{R}^n$ and if*

$$\mathbf{M}([du_k]) \le C \qquad \textit{and} \qquad u_k \to u \textit{ in } L^1_{loc}$$

*then $u \in MA(\Omega)$ and $\mathbf{M}([du]) \le \limsup_k \mathbf{M}([du_k])$. Moreover, for every $\alpha, \beta$ such that $|\alpha| + |\beta| = n$,*

$$M^{\bar\alpha\beta}(D^2 u_k) \rightharpoonup M^{\bar\alpha\beta}(D^2 u) \qquad \textit{weakly as measures.}$$

The corollary implies that if $\{u_k\}$ is a family of $C^\infty$ functions such that $\|M^{\bar\alpha\beta}(D^2u_k)\|_{L^1} \le C$ for every $\alpha, \beta$, then any $L^1_{loc}$ limit of a convergent subsequence is Monge-Ampère. (Such a sequence is precompact in $W^{1,p}_{loc}$ for $1 < p < n/n-1$, so we could equally well assume convergence in these spaces.)

## 2. PIECEWISE LINEAR FUNCTIONS

We say that a function $u$ is a piecewise linear function on a bounded set if the domain $\Omega \subset \mathbb{R}^n$ of $u$ is a *finite* union $\Omega = \cup_{i=1}^N F_i^n$, where $\{F_i^n\}$ is a collection of closed bounded polygons with open, pairwise disjoint interiors; and if $u \in W^{1,\infty}(\Omega)$, with $Du$ equal to a constant, say $Du = p_i$, on the interior of each $F_i^n$. In this section we prove that every such function is Monge-Ampère. This will be the basis for some examples we construct in later sections. As remarked earlier, this is certainly known to experts, but we hope that the proof gives some insight into the rather abstract general theory. In the piecewise linear setting, the geometric measure theory conditions of the definition of Monge-Ampère functions become much more concrete. In particular, in this section, all currents that appear are just finite linear combinations of (oriented) polygons.

We will write $\{F_i^k\}_{i=1}^{N(k)}$ to denote the set of $k$-faces of the polygons $\{F_i^n\}$, where a $k$-face is defined as a $k$-dimensional polygon that is the intersection of 2 or more $k+1$-faces, or equivalently of $n-k$ or more polygons $F_i^n$.

We fix an orientation for each $F_i^k$; we will below specify this further as convenient. We will not distinguish in our notation between the oriented polygon $F_i^k$ and the integral current corresponding to integration over $F_i^k$.

We will say that a piecewise linear function is *generic* if each $n-k$-face is the intersection of exactly $k+1$ polygons. The main result of this section is

**Lemma 2.1.** *Suppose that $u$ is a piecewise linear function on $\Omega \subset \mathbb{R}^n$. Then $u$ is Monge-Ampère, and for $k = 0, \ldots, n$ there exist polygonal $k$-currents*

$\{P_i^k\}_{i=1}^{N(n-k)}$ *such that*

$$[du] = \sum_{k=0}^{n} \sum_{i=1}^{N(n-k)} F_i^{n-k} \times P_i^k. \tag{2.1}$$

*If $u$ is a generic piecewise linear function, then each $P_i^k$ is a $k$-dimensional simplex with vertices $p_{\alpha_0}, \ldots, p_{\alpha_k}$, where $\alpha_0, \ldots, \alpha_k$ are such that $F_i^{n-k}$ is a shared $(n-k)$-face of $F_{\alpha_0}^n, \ldots F_{\alpha_k}^n$. In particular, if $u$ is generic then*

$$\mathbf{M}([du]) = \mathcal{L}^n(\Omega) + \sum_{k=1}^{n} \sum_{i=1}^{N(n-k)} \mathcal{H}^{n-k}(F_i^{n-k}) \ J_k(F_i^{n-k}) \tag{2.2}$$

*where $J_k(F_i^{n-k})$ denotes the $k$-dimensional jump of $Du$ along $F_i^{n-k}$, defined by*

$$J_k(F_i^{n-k}) = \mathcal{H}^k(\mathrm{co}(p_{\alpha_0}, \ldots p_{\alpha_k})), \quad F_i^{n-k} \text{ is a } (n-k)\text{-face of } F_{\alpha_0}^n, \ldots, F_{\alpha_k}^n,$$

*and* $\mathrm{co}(\cdots)$ *denotes the convex hull.*

If $u$ is not generic, it follows from (2.1) that

$$\mathbf{M}([du]) = \sum_{k=0}^{n} \sum_{i=1}^{N(n-k)} \mathcal{H}^{n-k}(F_i^{n-k}) \mathbf{M}(P_i^k)$$

but we do not know any easy formula for $\mathbf{M}(P_i^k)$ except in the generic case, where $\mathbf{M}(P_i^k)$ is exactly what we have called $J_k(F_i^{n-k})$.

We point out that by inspection of the proof below one can see the way in which, in this simple context at least, the Lagrangian condition enforces the uniqueness of $[du]$. The same mechanism is the main point in the more difficult proof of the basic uniqueness result, Theorem 1.1.

*Proof.* For every $k \geq 1$ and every $i$, the boundary of $F_i^k$ is a union of $k-1$ faces, so there are constants $\sigma_{i,j}^k \in \{0, \pm 1\}$ such that

$$\partial F_i^k = \sum_{j=1}^{N(k-1)} \sigma_{i,j}^k F_j^{k-1}. \tag{2.3}$$

Note that $\sigma_{i,j}^k \neq 0$ if and only if $F_j^{k-1}$ is a $k-1$-face of $F_i^k$.

To prove the lemma, we will show that there exists a current of the form (2.1) with all the properties required of $[du]$. We first translate these properties into conditions on the currents $\{P_i^k\}$. By using (2.3), one checks that for a current $T$ of the form (2.1),

$$\partial T = \sum_{k=0}^{n} \sum_{j=1}^{N(n-k)} F_j^{n-k} \times \left( (-1)^{n-k} \partial P_j^k + \sum_{i=1}^{N(n-k+1)} \sigma_{i,j}^{n-k+1} P_i^{k-1} \right)$$

Thus $\partial T = 0$ if and only if

$$\partial P^k_j = (-1)^{n-k+1} \sum_{i=1}^{N(n-k+1)} \sigma^{n-k+1}_{i,j} P^{k-1}_i \tag{2.4}$$

for all $k, j$. One can further verify from the definition (1.8) that a current of the form (2.1) is Lagrangian if and only if

$$TF^{n-k}_i = (TP^k_i)^\perp \qquad \text{for all } i, k \text{ such that } P^k_i \neq 0 \tag{2.5}$$

where $TF^{n-k}_i, TP^k_i$ denote the tangent spaces to $F^{n-k}_i$, $P^k_i$ respectively. (Strictly speaking these tangent spaces belong to different copies of $\mathbb{R}^n$; for simplicity, throughout the proof we identify these in the canonical way.) Finally, it follows from the definitions that a current $T$ of the form (2.1) satisfies (1.6) if and only if

$$P^0_i = [p_i], \qquad i = 1, \dots, N(n), \tag{2.6}$$

where we recall that $p_i$ is the constant value of $Du$ on $F^n_i$. (Here we are implicitly taking the polygons $F^n_i$ to be oriented in the standard fashion.)

So we must show that we can select polygons $\{P^k_i\}$ with finite mass that satisfy (2.4), (2.5), and (2.6). Our proof will also show that

$$P^k_i \subset \text{the plane generated by } \{p_{\alpha_j} \ : F^{n-k}_i \text{ is a face of } F^n_{\alpha_j}\} \tag{2.7}$$

for every $k \in \{0, \dots, n\}$ and $i \in \{0, \dots N(n-k)$. Indeed, we will see that this is more or less equivalent in this setting to the Lagrangian condition (2.5). Here the plane generated by a finite set $\{v_0, \dots, v_N\}$ is just the set $\{v_0 + \sum_{i=1}^N a_i(v_i - v_0) : a_i \in \mathbb{R}\}$. From (2.6) we note that (2.7) is trivially true when $k = 0$.

Since $\{P^0_i\}_{i=0}^{N(n)}$ are already determined by (2.6), we can use (2.4) and (2.5) to determine $\{P^1_i\}_{i=1}^{N(n-1)}$. To do this, fix $j \in \{1, \dots, N(n-1)\}$, and note that the $(n-1)$-face $F^{n-1}_j$ is the intersection of exactly two simplices, say $F^n_{i_1}, F^n_{i_2}$. That is, $\sigma^n_{i,j} \neq 0$ if and only if $i = i_1$ or $i_2$. Moreover, the set $F^n_{i_1} \cap F^n_{i_2}$ inherits opposite orientations from $F^n_{i_1}$ and $F^n_{i_2}$; exactly one of these must agree with the orientation of $F^{n-1}_j$. Thus (2.4) reduces (in view of (2.6)) to

$$\partial P^1_j = \pm([p_{i_1}] - [p_{i_2}]).$$

Thus (2.4) is satisfied if we specify that $P^1_j = \pm[p_{i_1}, p_{i_2}]$. Moreover, because $u$ is continuous across $F^{n-1}_j$, it follows that $(p_{i_1} - p_{i_2}) \cdot \tau = 0$ for every vector $\tau$ tangent to $F^{n-1}_j$, which shows that our choice of $P^1_j$ satisfies (2.5). Note also that (2.7) holds.

Now for $k \geq 2$ we assume by induction that we have found $\{P^\ell_i\}_{i=1}^{N(n-\ell)}$ with the required properties, for $\ell = 0, \dots, k-1$, and we show that the system of equations (2.4) with constraints (2.5) can be solved, and that the solutions satisfy (2.7). We first check that currents $P^k_j$ solving (2.4) exist. To do this we need to check that the right-hand side of (2.4) is a boundary. Using the induction hypothesis,

$$\partial\left( \sum_{i=1}^{N(n-k+1)} \sigma^{n-k+1}_{i,j} P^{k-1}_i \right) = \sum_{i_1=1}^{N(n-k+1)} \sum_{i_2=1}^{N(n-k+2)} \sigma^{n-k+1}_{i_1,j} \sigma^{n-k+2}_{i_2,i_1} P^{k-2}_{i_2}. \tag{2.8}$$

However,

$$0 = \partial^2 F^k_i = \partial \sum_{j_1} \sigma^k_{i,j_1} F^{k-1}_{j_1} = \sum_{j_1,j_2} \sigma^k_{i,j_1} \sigma^{k-1}_{j_1,j_2} F^{k-1}_{j_2}$$

Thus

$$\sum_{j_1} \sigma^k_{i,j_1} \sigma^{k-1}_{j_1,j_2} = 0$$

for every $i, j_2$, which implies that the right-hand side of (2.8) vanishes, as desired.

It follows that $P^k_j$ solving (2.4) exist for every $j \in \{1, \ldots, N(n-k)\}$. We now show that we can find such a current satisfying in addition the Lagrangian condition. To see this, observe that by induction, (2.7) holds for every $P^{k-1}_i$, so that the right-hand side of (2.4) is supported in the

$$\text{plane generated by } \{p_{\alpha_\ell} \ : F^{n-k}_j \text{ is a face of } F^n_{\alpha_\ell}\}$$

Therefore we can find $P^k_j$ solving (2.4) and supported in the same plane. We may assume that $P^k_j$ is nonzero, as otherwise (2.5) is vacuous. Then clearly$TP^k_j$ is contained in the tangent space to this plane, which is

$$\text{span}\{p_{\alpha_\ell} - p_{\alpha'_\ell} \ : F^{n-k}_j \text{ is a face of both } F^n_{\alpha_\ell} \text{ and } F^n_{\alpha'_\ell}\}.$$

And since $u$ is continuous, if $\tau$ is any vector tangent to $F^{n-k}_j$, then $(p_{\alpha_\ell} - p_{\alpha'_\ell}) \cdot \tau = 0$ when $F^{n-k}_j$ is a face of both $F^n_{\alpha_\ell}$ and $F^n_{\alpha'_\ell}$, exactly as in the $k = 1$ case. Thus $TP^k_j \subset (TF^{n-k}_j)^\perp$. The opposite inclusion follows from noting that $TP^k_j$ is $k$-dimensional, since $P^k_i$ is nonzero, and $TF^{n-k}_j$ is $n-k$-dimensional.

In the generic case we can write down an explicit formula for $[du]$. This necessitates a bit more notation. Suppose that $F^n_{i_0}, \ldots, F^n_{i_k}$ are $n$-faces that share a common $(n-k)$-face. This common face will be denoted $F^{n-k}_{i_0 \ldots i_k}$, the orientation of which is taken to depend on the order in which the indices $i_0 \ldots i_k$ are listed, in such a way that

$$\partial F^{n-k}_{i_0 \ldots i_k} = (-1)^n \sum_{i_{k+1}} F^{n-k-1}_{i_0 \ldots i_k i_{k+1}}. \tag{2.9}$$

where we use the convention that $F^{n-j}_{i_0 \ldots i_j} = 0$ if $F^n_{i_0}, \ldots, F^n_{i_j}$ do not share a common $(n-j)$-face. One can check that if $\sigma$ is any permutation of $k+1$

elements, then

$$F^{n-k}_{i_0\dots i_k} = \mathrm{sign}(\sigma)\, F^{n-k}_{\sigma(i_0)\dots\sigma(i_k)}. \tag{2.10}$$

Next, for $p_0, \dots, p_k \in \mathbb{R}^n$ with $0 \le k \le n$, we write $[p_0, \dots, p_k]$ to denote the $k$-dimensional simplex with vertices at the given points. We stipulate that such a simplex is oriented as as described in Federer [2] 4.1.11 which implies in particular that a 0-simplex $[p]$ is taken with the standard orientation, and that

$$\partial[p_0, \dots, p_k] = \sum_{i=0}^{k} (-1)^i [p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_k]. \tag{2.11}$$

We claim that if $u$ is a generic piecewise linear function, then

$$[du] = \sum_{k=0}^{n} \sum_{i_0<\dots<i_k} F^{n-k}_{i_0\dots i_k} \times [p_{i_0}, \dots, p_{i_k}]. \tag{2.12}$$

Our earlier arguments show that (1.6) and (1.4), which as above reduce to (2.6) and (2.5), are satisfied. So we only need to check that the current on the right-hand side above has vanishing boundary. To do this, apply the operator $\partial$ to the right-hand side of (2.12), and then collect all terms containing some fixed $F^{n-k-1}_{i_0\dots i_{k+1}}$. Then the terms of the form $\partial F^{n-k}_{\dots} \times [\cdots]$ and those of the form $(-1)^{n-k} F^{n-k}_{\dots} \times \partial[\cdots]$ can be see from (2.9), (2.10), and (2.11) to cancel.

The remaining claim (2.2) follows from the explicit formula (2.12), since it is clear that the terms on the right-hand side of (2.12) are pairwise disjoint, and that

$$\mathbf{M}(F^{n-k}_{i_0\dots i_k} \times [p_{i_0}, \dots, p_{i_k}]) = \mathcal{H}^{n-k}(F^{n-k}_{i_0\dots i_k})\mathcal{H}^k([p_{i_0}, \dots, p_{i_k}]),$$

and that $\mathcal{H}^k([p_{i_0}, \dots, p_{i_k}]) = J_k(F^{n-k}_{i_0\dots i_k})$ as defined in the statement of the theorem. □

## 3. REGULARITY

Regularity properties of Monge-Ampère functions are not well-understood. It is rather easy to see that if $u$ is a Monge-Ampère function, then the second derivatives of $u$ are measures[1] and it is also easy to construct examples of Monge-Ampère functions — for example, the piecewise affine functions of the last section — that do not belong to $W^{2,p}$ for any $p \ge 1$. Beyond these statements, not much is known.

[1] Note that in (1.13) the we have already defined the $1\times 1$ minors $M^{ij}(D^2u)$ as measures on $\Omega \times \mathbb{R}^n$. The point here is that if we let $u_{x_ix_j}$ denote the marginal on $\Omega$ of $M^{ij}(D^2u)$, ie $u_{x_ix_j}(A) := M^{ij}(D^2u)(A\times\mathbb{R}^n)$ whenever the latter is well-defined, then one can verify that $(u_{x_ix_j})_{j=1}^n$ are the distributional derivatives of $u_{x_i}$, using the condition $\partial[du] = 0$. For a full proof see Fu [4]

In particular, when $n \geq 3$ it is not known whether Monge-Ampère functions on $\mathbb{R}^n$ are necessarily continuous, although this seems overwhelmingly likely For $n \leq 2$, continuity follows from the fact that the second derivatives are measures.

In this section we show that Monge-Ampère functions can fail to be $C^{0,\alpha}$ when $\alpha > \frac{2}{n+1}$. We also give a lemma, which is no doubt classical, showing that if $u$ is a function that is convex on its support, then

$$[u]_{C^{0.1/n}} \leq C(\text{diam supp } u)\| \det D^2 u\|_{L^1}^{1/n},$$

and we show that no estimate of this sort can hold for $\alpha > 1/n$.

**Lemma 3.1.** *If $\Omega$ is a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$, then there exists $u \in MA(\Omega)$ with compact support in $\Omega$ and such that*

$$u \notin C^{0,\alpha}(\Omega) \quad \textit{for any } \alpha > \frac{2}{n+1}.$$

*Proof.* For concreteness we take $\Omega := \{x \in \mathbb{R}^n : |x| < 2\}$.

**1**. We first show that for $\varepsilon \in (0,1]$ and $q > 0$, there exists a function $u_{\varepsilon,q} \in MA(\Omega)$ such that

$$\mathbf{M}([du_{\varepsilon,q}]) - \mathcal{L}^n(\Omega) \leq C\varepsilon^{n-1-q}, \qquad [u_{\varepsilon,q}]_{C^{0,\alpha}} \geq c\varepsilon^{2-\gamma(2+q)} \tag{3.1}$$

where the constants $C, c$ are independent of $\varepsilon, q$. Indeed, we define

$$u_{\varepsilon,q}(x) = \Big(\min\{\varepsilon x_1, \ldots, \varepsilon x_n, \varepsilon^{-q}(\varepsilon - \sum x_i)\}\Big)^+$$

where $a^+ = \max\{a, 0\}$. Note that $u_{\varepsilon,q}$ is supported in

$$S_\varepsilon := \{(x_1, \ldots, x_n) : x_i \geq 0 \ \ \forall i, \ \sum x_i \leq \varepsilon\}, \tag{3.2}$$

and in particular has compact support in $\Omega$.

We first estimate $\mathbf{M}([du_{\varepsilon,q}])$ using (2.2) from the previous section. Note that $u_{\varepsilon,q}$ is what we have called a generic piecewise linear function, so this result is applicable. Let us write $p_0 := -\varepsilon^{-q}(1, \ldots, 1)$, $p_i = \varepsilon e_i$ for $i = 1, \ldots, n$, and $p_{n+1} = 0$, where $e_i$ denotes the standard unit vector. Then the values assumed by $Du_{\varepsilon,q}$ are $\{p_0, \ldots, p_n, p_{n+1}\}$. If $k \geq 1$, then any simplex $[p_{\alpha_0}, \ldots, p_{\alpha_k}]$ has its $k$-dimensional volume bounded by $\prod_{j=1}^k |p_{\alpha_j} - p_{\alpha_0}|$. By permuting the $p_{\alpha_i}$s, we can assume that $\alpha_0 \neq 0$, which implies that $|p_{\alpha_j} - p_{\alpha_0}| \leq C\varepsilon$ except for at most 1 choice of $j$, and $|p_{\alpha_j} - p_{\alpha_0}| \leq C\varepsilon^{-q}$ for all $j$. Thus

$$\mathcal{H}^k([p_{\alpha_0}, \ldots, p_{\alpha_k}]) \leq C\varepsilon^{k-1-q}.$$

Also, if $F_i^{n-k}$ is an $n-k$ face of the triangulation associated with $u_{\varepsilon,q}$, then $F^{n-k}$ is contained in a ball of size $C\varepsilon$, and so $\mathcal{H}^{n-k}(F_i^{n-k}) \leq C\varepsilon^{n-k}$. From these considerations and (2.2) we conclude that

$$\mathbf{M}([du_{\varepsilon,q}]) \leq C\varepsilon^{n-1-q} + \mathcal{L}^n(\Omega).$$

Let $x_{\varepsilon,q}$ be the point at which $\varepsilon x_1 = \ldots = \varepsilon x_n = \varepsilon^{-q}(\varepsilon - \sum x_i)$; this is the unique point at which $u_{\varepsilon,q}$ attains its maximum. Also, let $y_{\varepsilon,q} = \frac{\varepsilon}{n}(1,\ldots,1)$ One can check that

$$x_{\varepsilon,q} = \frac{\varepsilon}{n+\varepsilon^{1+q}}(1,\ldots,1) = y_{\varepsilon,q} - \frac{\varepsilon^{q+2}}{n^2}(1,\ldots,1) + O(\varepsilon^{2q+3})$$

and that

$$u_{\varepsilon,q}(x_{\varepsilon,q}) = \frac{\varepsilon^2}{n+\varepsilon^{1+q}} = \frac{\varepsilon^2}{n} + O(\varepsilon^{q+3}).$$

Thus for any $\gamma \in (0,1]$, since $u_{\varepsilon,q}(y_{\varepsilon,q}) = 0$,

$$[u_{\varepsilon,q}]_{C^{0,\gamma}} \geq \frac{|u_{\varepsilon,q}(x_{\varepsilon,q}) - u_{\varepsilon,q}(y_{\varepsilon,q})|}{|x_{\varepsilon,q} - y_{\varepsilon,q}|^\gamma} \approx \varepsilon^{2-\gamma(2+q)}.$$

Thus we have proved (3.1).

**2**. Now for $j \geq 1$, let $u_j(x) = u_{\varepsilon_j,q_j}(x - 2^{-j}e_1)$ where $\varepsilon_j = 2^{-j^2}$ and $q_j = n - 1 - \frac{1}{j}$. Here $e_1$ is the standard unit vector $(1,0,\ldots,0)$.

Then define $U_k(x) = \sum_{j=1}^k u_j(x)$, and finally let $u = \lim_{k\to\infty} U_k$. (It is easy to see that $U_k(x)$ converges in $L^1$ for example as $k \to \infty$.)

Observe from (3.2) that the supports of $u_j, u_{j'}$ are separated by a positive distance if $j \neq j'$, so that $U_k$ is a generic piecewise linear function for every $k$. This disjoint support property and (2.2) further imply that

$$\mathbf{M}([dU_k]) - \mathcal{L}^n(\Omega) = \sum_{j=1}^k [\mathbf{M}([du_j] - \mathcal{L}^n(\Omega))]$$

It then follows from (3.1) that

$$\mathbf{M}([dU_k]) \leq \mathcal{L}^n(\Omega) + C\sum_{j=1}^k (2^{-j^2})^{1/j} \leq C$$

where the constant is independent of $k$.

In particular, since $U_k \to u$ in $L^1$ and $\mathbf{M}([dU_k]) \leq C$, it follows from Corollary 1.1 that $u$ is a Monge-Ampère function.

Now define $x_j = x_{\varepsilon_j,q_j} + 2^{-j}e_1$, $y_j = y_{\varepsilon_j,q_j} + 2^{-j}e_1$, (using notation from step 1 above) so that

$$|x_j - y_j| \leq C\varepsilon_j^{q_j+2}, \qquad |u_j(x_j) - u_j(y_j)| \geq c\varepsilon_j^2.$$

Note also that $u(x_j) = u_j(x_j)$ and $u(y_j) = u_j(y_j)$. Then for any $\gamma > 2/(n+1)$,

$$[u]_{C^{0,\gamma}} \geq \frac{|u(x_j) - u(y_j)|}{|x_j - y_j|^\gamma} \geq c\varepsilon_j^{2-\gamma(q_j+2)} \to \infty$$

as $j \to \infty$. Thus $u \notin C^{0,\gamma}(\Omega)$ for $\gamma > 2/(n+1)$.

□

The above lemma naturally raises the question of what sort of continuity properties follow from the condition $\mathbf{M}([du]) < \infty$. This is related to (but probably harder than) the question of what continuity properties are implied by $L^1$ bounds on $\det D^2u$. In the next lemma we recall that this easier question is more or less completely understood in the convex case. This is presumably classical, although we do not know a reference for it. In this lemma, unlike in the rest of this paper, $\operatorname{Det} D^2u$ is understood in the sense of Alexandrov as a measure on $\Omega$, so that for a Borel set $A \subset \Omega$,

$$\operatorname{Det} D^2u(A) = \mathcal{L}^n(\{p : p \in \partial u(x) \text{ for some } x \in A\}),$$

where $\partial u(x)$ denotes the subgradient of $u$ at $x$. In particular, if $u \in W^{2,n}(\Omega)$ then $\operatorname{Det} D^2u(A) = \int_A \det D^2u(x)\,dx$.

**Lemma 3.2.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex open set and that $u : \bar{\Omega} \to \mathbb{R}$ is a continuous convex function with $u = 0$ on $\partial\Omega$. If in addition $\operatorname{Det} D^2u(\Omega) < \infty$, then $u \in C^{0,1/n}(\bar{\Omega})$, and*

$$[u]_{C^{0,1/n}} \le C \left[\operatorname{Det} D^2u(\Omega)\right]^{1/n} \tag{3.3}$$

*where the constant $C$ depends only on the diameter of $\Omega$.*

*Conversely, there exists a convex $\Omega \subset \mathbb{R}^n$ and a function $u : \Omega \to \mathbb{R}$ which is convex in $\Omega$ and vanishing on $\partial\Omega$, such that $\operatorname{Det} D^2u(\Omega) < \infty$ and*

$$u \notin C^{0,\gamma}(\bar{\Omega}) \qquad \text{for any } \gamma > \frac{1}{n}. \tag{3.4}$$

The (perhaps limited) relevance of this lemma to our main concerns is as follows: first, if $u \in MA_{loc}(\Omega)$ and $u$ vanishes on $\partial\Omega$, then (3.3) suggests that one might hope that $u \in C^{0,1/n}$ and that $[u]_{C^{0,1/n}} \le C(\Omega)|\operatorname{Det} D^2u|(\Omega \times \mathbb{R}^n)$, where $\operatorname{Det} D^2u$ is now understood in the sense of Monge-Ampère functions, as a measure in $\Omega \times \mathbb{R}^n$ as in (1.14), and $|\operatorname{Det} D^2u|$ denotes the associated total variation measure.

Second, (3.4) shows that, if it is true that Monge-Ampère functions are $C^{0,\alpha}$ for some $\alpha > 1/n$, then any proof of this fact must exploit not only control over $|\operatorname{Det} D^2u|$, but must also use some other information encoded in $[du]$, such as bounds on second derivatives of $u$ or on $k \times k$ minors of $D^2u$ for some $k < n$.

We now give the

*proof of Lemma 3.2.* **1**. We first prove (3.3), using classical ideas of Alexandrov to argue that if $[u]_{C^{0,1/n}}$ is large then necessarily the image of the subgradient map has large measure.

We may assume after rescaling that the diameter of $\Omega$ is at most 1. We must estimate $|u(y) - u(z)|$ for $y, z \in \bar{\Omega}$. Initially, let us assume for simplicity that $z \in \partial\Omega$, so that $u(y) \le u(z) = 0$. Let $z' \in \partial\Omega$ be a point such that $|y-z'| = \mathrm{dist}(y, \partial\Omega) =: d$. In particular $u(z') = u(z) = 0$ and $|y-z'| \le |y-z|$. We may assume after a translation that $z' = 0$ and $y = (d, 0, \ldots, 0)$. Then, since $\Omega$ is convex and has diameter less than 1, and by the choice of $z'$,

$$\Omega \subset \{x : 0 < x_1 < 1, |x_j| < 1 \text{ for } j = 2, \ldots, n\} =: R.$$

The point is that, since $z'$ minimizes the distance from $\partial\Omega$ to $y$, the line joining $y$ to $z'$ must be normal to a supporting hyperplane at $z'$.

We write $a := u(y)$, and we define the linear functions

$$\ell_i^{\pm}(x) = a(1 \pm x_i) \qquad i = 2, \ldots, n$$

and

$$\ell_1^+ = a\frac{x_1}{d}, \qquad \ell_1^- = \frac{a(1-x_1)}{1-d}.$$

Then $\ell_i^+(y) = \ell_i^-(y) = a$ for all $i$. Also, since $a \le 0$, $\ell_i^+(x) \le 0$ and $\ell_i^-(x) \le 0$ for all $x$ in the rectangle $R$, and hence in $\Omega$, for all $i$.

Thus if $\ell$ is any convex combination of the above linear functions, then $\ell(y) = a = u(y)$, and $\ell(x) \le 0 = u(x)$ on $\partial\Omega'$. It follows that $\ell - u$ attains its maximum at some point $x_0 \in \Omega$. In other words, $\ell(x) - u(x) \ge \ell(x_0) - u(x_0)$ for all $x \in \Omega$, which says exactly that $D\ell \in \partial u(x)$.

If we let $p_i^{\pm}$ denote the (constant) gradient of $\ell_i^{\pm}$, it follows that every convex combination of $\{p_i^{\pm}\}$ belongs to $\partial u(x)$ for some $x \in \Omega$. It is easy to check that the convex hull $\mathrm{co}\{p_i^{\pm}\}$ has measure greater than $c|a|^n/d = c|u(y)|^n/d$. Thus

$$\mathrm{Det}\, D^2u(\Omega) \ \ge \ \mathcal{L}^n(\mathrm{co}\{p_i^{\pm}\}) = c\frac{|u(y)|^n}{|y-z'|} \ge c\frac{|u(y)-u(z)|^n}{|y-z|}$$

since $u(z) = 0$ and $|y-z| \ge |y-z'|$.

We finally claim that the same estimate holds if $z \notin \partial\Omega$. Indeed, if $u(y) \le u(z) < 0$, we define $\Omega' := \{x \in \Omega : u(x) < u(z)\}$, and for $x \in \bar\Omega'$ we define $u'(x) = u(x) - u(z)$. Then the above arguments apply to $u'$ on $\Omega'$ and yield the estimate $\mathrm{Det}\, D^2u(\Omega) \ge \mathrm{Det}\, D^2u(\Omega') \ge c\frac{|u(y)-u(z)|^n}{|y-z|}$. Thus we can take the supremum over all $y, z \in \Omega$ to complete the proof of the (3.3).

**2**. We now construct a convex function that is not $C^{0,\gamma}$ for any $\gamma > 1/n$, but such that $\mathrm{Det}\, D^2u$ is a finite measure. Let $f : [0,1] \to \mathbb{R}$ denote a strictly convex function such that $f(0) = f(1) = 0$, and to be further specified later. We will write points in $\mathbb{R}^n$ in the form $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$. Let $\Omega := \{(x', x_n) \ : 0 < x_n \text{ and } x_n + |x'| < 1\}$, and define

$$u(x', x_n) = (1-|x'|)f(\frac{x_n}{1-|x'|}) \qquad \text{for } (x', x_n) \in \Omega.$$

We first prove that $u$ is convex. To do this, for $p \in \mathbb{R}^{n-1}$ and $s \in [0,1]$ define

$$u_{p,s}(x', x_n) = f(s) + f'(s)(x_n - s) - x' \cdot p[f(s) - sf'(s)].$$

We claim that

$$u(x) := \max_{|p|\le 1, s\in[0,1]} u_{p,s}(x) \qquad \text{for } x \in \Omega. \tag{3.5}$$

This will imply in particular that $u$ is convex. Note also that, since $f$ is convex, $0 = f(0) \ge f(s) - sf'(s)$ for all $s \in [0,1]$, which implies that

$$\begin{aligned}\max_{|p|\le 1, s\in[0,1]} u_{p,s}(x) &= \max_{s\in[0,1]} \left[ f(s) + f'(s)(x_n - s) + \max_{|p|\le 1}(x'\cdot p)\,(sf'(s) - f(s)) \right] \\ &= \max_{s\in[0,1]} [f(s)(1-|x'|) + f'(s)(x_n - s(1-|x'|)] \\ &= (1-|x'|) \max_{s\in[0,1]} \left[ f(s) + f'(s)(\frac{x_n}{1-|x'|} - s) \right].\end{aligned}$$

Since $f$ is convex, (3.5) follows.

Next we claim that

$$\{\partial u(x) : x \in \Omega\} = \{(q', q_n) : q_n = f'(s), |q'| \le g(s) \text{ for some } s \in (0,1)\}. \tag{3.6}$$

where we write $g(s) = sf'(s) - f(s)$. The right-hand side of (3.6) is exactly $\{Du_{p,s} \ : |p| \le 1, s \in (0,1)\}$. For every $|p| \le 1$ and every $s$, $u_{p,s}(0,s) = f(s) = u(0,s)$. Then (3.5) implies that $u_{p,s}$ is a supporting hyperplane at $(0,s)$, which implies the inclusion $\supset$ in (3.6).

To prove the other inclusion, suppose that $q \in \partial u(x)$ for some $x = (x', x_n)$. If $x' \ne 0$, then $u$ is smooth and $\partial u(x) = \{Du(x)\} = \{Du_{p,s}\}$ for $p = x'/|x'|$ and $s = x_n/(1-|x'|)$, by a short calculation. And if $q = (q', q_n)$ belongs to $\partial u(x)$ for $x = (0, x_n)$, then

$$(1-|y'|)f(\frac{y_n}{1-|y'|}) \ \ge f(x_n) + q'\cdot y' + q_n(y_n - x_n) \tag{3.7}$$

for all $y \in \Omega$. Taking $y$ of the form $y = (0, y_n)$, we find that $q_n = f'(x_n)$. And considering $y$ of the form $(y', (1-|y'|)x_n)$, we find after some calculations that $q' \cdot \frac{y'}{|y'|} \le g(x_n)$. This completes the proof of the inclusion $\subset$ in (3.6).

We will eventually take $f$ to be strictly convex and $C^2$ on $(0,1]$, so that $f'$ is invertible and $f'(1)$ makes sense. Then we deduce from (3.6) that

$$\{\partial u(x) : \ x \in \Omega\} = \{(q', q_n) : \ f'(0) < q_n < f'(1), |q'| \le g(f'^{-1}(q_n))\}.$$

(By $f'(0)$ we mean $\lim_{s\searrow 0} f'(s)$, which will turn out to be $-\infty$ for us.) Let us write $h$ for $f'^{-1}$. From the above we deduce that

$$\operatorname{Det} D^2u(\Omega) = c_n \int_{f'(0)}^{f'(1)} (g\circ h(t))^{n-1}\, dt \ = \ c_n \int_0^1 g(s)^{n-1} f''(s)\, ds$$

where we have made the change of variables $f'(s) = t$, so that $h(t) = s$. Observe also that $f''(s) = \frac{1}{s}g'(s)$, so

$$\text{Det}\, D^2 u(\Omega) \;=\; c_n \int_0^1 g(s)^{n-1} g'(s) \frac{ds}{s} \;=\; \frac{c_n}{n} \int_0^1 \frac{d}{ds}[g(s)^n] \frac{ds}{s}. \tag{3.8}$$

Finally, we take $f$ so be a strictly convex function, smooth in $(0, 1]$, such that

$$f(s) = -\frac{s^{1/n}}{|\log s|} \qquad \text{for } 0 < s \le \frac{1}{2}$$

and such that $f(1) = 0$ and $f', f''$ are bounded on $[1/2, 1]$. (One can verify by differentiating that $s^{1/n}(\log s)^{-1}$ is convex for $s \in (0, 1)$.)

Since $u(0, x_n) = f(x_n)$ it is clear that (3.4) holds. To finish the proof we must therefore check that $\text{Det}\, D^2 u(\Omega)$ is finite. Because $f$ is smooth away from $s = 0$, we see from (3.8) that it suffices to check that

$$\int_0^{1/2} g(s)^{n-1} g'(s) \frac{1}{s} ds < \infty.$$

To do this, we check by explicit computations that for $0 < s \le 1/2$, there exists $C$ such that

$$0 < g(s) \le -C f(s), \qquad 0 < g'(s) \le -C \frac{f(s)}{s}.$$

Thus

$$\int_0^{1/2} g(s)^{n-1} g'(s) \frac{1}{s} ds \le C \int_0^{1/2} f(s)^n \frac{1}{s^2} ds \;=\; C \int_0^{1/2} \frac{1}{s|\log s|^n}\, ds < \infty$$

since $n \ge 2$. □

*Remark* 1. One can construct a similar example that is smooth in $\Omega$ by taking $u$ of the form

$$u(x', x_n) = \rho(x') f(\frac{x_n}{\rho(x')}) \qquad \text{for } \rho(x') = 1 - |x'|^2 \text{ and } f \text{ as above.}$$

It is also not hard to modify $u$ to construct an example in a smooth domain, since the corners in $\partial\Omega$ occur far from the point $(0, 0)$ at which various Hölder seminorms blow up.

*Remark* 2. It is much easier to prove that no estimate of the form

$$[v]_{C^{0,\gamma}} \le |\,\text{Det}\, D^2 v|(\Omega) \qquad v \in C_0(\bar{\Omega}) \cap C^\infty(\Omega),\; u \text{ convex in } \bar{\Omega} \tag{3.9}$$

can hold for any $\gamma > 1/n$, where the subscript in $C_0$ indicates functions that vanish on $\partial\Omega$. Indeed, this follows from an easy scaling considerations. To see this, let $\Omega$ be a set of diameter at most 1 and let $v \in C_0(\bar{\Omega}) \cap C^\infty(\Omega)$ be a convex function. Suppose for concreteness that $0 \in \Omega$ with $v(0) < 0$, and that $(a, 0, \ldots, 0) \in \partial\Omega$, so that $v(a, 0, \ldots, 0) = 0$. Necessarily $a < 1$.

Fix a parameter $q > 0$ and let $\Omega_\varepsilon = \{(\varepsilon x_1, x_2, \dots, x_n\}) : x \in \Omega\}$. Define $v_{\varepsilon,q}(x_1, \dots, x_n) = \varepsilon^q v(x_1/\varepsilon, x_2, \dots, x_n)$. Then it is easy to check that

$$\det D^2 v_{\varepsilon,q}(x_1, \dots, x_n) = \varepsilon^{nq-2} \det D^2 v(x_1/\varepsilon, x_1, \dots, x_n),$$

and hence that

$$\int |\det D^2 v_{\varepsilon,p}| = \varepsilon^{nq-1} \int |\det D^2 v|.$$

Note also that

$$[v_{\varepsilon,q}]_{C^{0,\gamma}} \geq \frac{|v_{\varepsilon,q}(0) - v_{\varepsilon,q}(a\varepsilon, 0, \dots, 0)|}{(a\varepsilon)^\gamma} \ = \ \varepsilon^{q-\gamma} \frac{|v(0)|}{a^\gamma}.$$

This easily implies that (3.9) fails when $\gamma > 1/n$.

## 4. LOCAL REPRESENTATIONS OF $\operatorname{Det} D^2 u$

In this section, we write $\operatorname{Det} D^2 u$ for $u \in MA_{loc}(\Omega)$ to denote the measure in $\Omega \times \mathbb{R}^n$ as defined in (1.14).

It follows from general geometric measure theory considerations that, given a Monge-Ampère function $u$, there exists a function $d : \Omega \times \mathbb{R}^n \to \mathbb{Z}$, defined for *every* $x \in \Omega$ and $\mathcal{L}^n$ a.e. $\xi$, such that

$$\int \phi(x,\xi) \operatorname{Det} D^2 u(dx, d\xi) \ = \ \int_{\mathbb{R}^n} \left( \int_\Omega \phi(x,\xi) d(x,\xi) d\mathcal{H}^0(x) \right) \mathcal{L}^n(d\xi). \tag{4.1}$$

In general, $d(x,\xi)$ should reflect the local behavior of the level sets of

$$y \mapsto u(y) - \xi \cdot y =: u_\xi(y)$$

near $y = x$. Theorems of this sort have been established in the different but related context of normal cycles associated with certain subsets of $\mathbb{R}^n$, see for example [19], [6], [18], [7], [14]; this list is far from exhaustive. In these theorems, roughly speaking, $\operatorname{Det} D^2 u$ in (4.1) is replaced by a measure in $\mathbb{R}^n \times S^{n-1}$ associated with the Gaussian curvature and coming from an integral current in $\mathbb{R}^n \times S^{n-1}$ that roughly speaking corresponds to the (generalized) graph of the Gauss map of a possibly rather rough subset of $\mathbb{R}^n$. These results strongly suggest that for quite general Monge-Ampère functions, $d(x,\xi)$ in (4.1) should admit the characterization

$$\begin{aligned} d(x,\xi) = \lim_{r \searrow 0} \lim_{\varepsilon \searrow 0} [\chi( \{y : |y-x| < r, u_{x,\xi}(y) < \varepsilon\} ) \\ - \chi( \{y : |y-x| < r, u_{x,\xi}(y) < -\varepsilon\} )] \end{aligned} \tag{4.2}$$

where $\chi$ denotes the Euler characteristic and $u_{x,\xi}(y) = u(y) - [u(x) + \xi \cdot (y-x)]$. Note that when (4.2) holds, $d(x,\xi)$ is nonzero exactly when the local topology of level sets of $y \mapsto u(y) - \xi \cdot y$ changes at $y = x$.

The proofs of results about normal cycles cited above can almost surely be transposed to yield corresponding results about classes of Monge-Ampère functions. For example, [19], [6], [18] establish analogs of (4.1), (4.2) when the set underlying the normal cycle is a union of sets of positive reach. A set of positive reach is roughly analogous to a semiconvex function, and a union of such sets is by the same token analogous to a function of the form

$$u(x) = \inf\{u_i(x)\}_{i=1}^M \qquad \text{for } u_1, \dots, u_M \text{ semiconvex functions.} \tag{4.3}$$

(Fu [4] proves that functions $u$ of this form are locally Monge-Ampère.) Presumably (4.2) can be established for such functions following arguments[2] developed in [6], [18] or other references.

In any case, our modest goal in this section is simply to illustrate (4.1), (4.2) by giving a reasonably elementary proof in a simple but nontrivial situation, that of homogeneous functions on $\mathbb{R}^2$. We will prove

**Proposition 4.1.** *Suppose that $u \in MA_{loc}(\mathbb{R}^2)$ is homogeneous of degree 1 Then(4.1) holds with*

(4.4)
$$d(x,\xi) = \begin{cases} 1 - \frac{1}{2}\#[\textit{sign changes of } (u(x) - \xi \cdot x)|_{S^1}] & \mathcal{L}^2 \textit{ a.e. } \xi \quad \textit{if } x = 0, \\ 0 & \textit{if not.} \end{cases}$$

Note that (4.4) is consistent with (4.2).

We will write 2-forms in $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$\psi = \psi_0 dx^1 \wedge dx^2 + \psi_1^{ij} dx^i \wedge d\xi^j + \psi_2 d\xi^1 \wedge d\xi^2 \tag{4.5}$$

The main part of the proof of Proposition 4.1 is contained in the following

**Lemma 4.1.** *Suppose that $u$ is homogeneous of degree 1 and smooth away from the origin. Then $u$ is Monge-Ampere. Moreover, for $\psi$ of the form (4.5),*

$$[du](\psi) = \int_{\mathbb{R}^2} [\psi_0(x, Du(x)) dx^1 \wedge dx^2 + \psi_1^{ij}(x, Du) dx^i \wedge d(u_{x_j})]$$
$$+ \int_{\mathbb{R}^2} \psi_2(0,\xi) \tilde{d}_0(\xi) \mathcal{L}^2(d\xi) \tag{4.6}$$

*where*

$$\tilde{d}_0(\xi) := \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma - \xi}{|\gamma - \xi|} \wedge \frac{d}{d\theta}\left(\frac{\gamma - \xi}{|\gamma - \xi|}\right) d\theta, \quad \gamma(\theta) := Du(re^{i\theta}) \tag{4.7}$$

[2]The basic idea in many of these proofs is to establish the result first for $C^{1,1}$ sets; then to use an approximation argument to deduce that it still holds for some rougher class of sets, such as sets of positive reach; and finally to use additivity properties of the Euler characteristic to extend the result to a larger class of sets such as those generated by suitable unions of sets of positive reach. Some aspects of this line of argument are easier in the case of Monge-Ampère functions than in the geometric setting.

*for* $\xi \in \mathbb{R}^2 \setminus Image(\gamma)$*. Finally, if* $K \subset \mathbb{R}^2$ *is any compact set then*

$$\|[du]\|(K \times \mathbb{R}^2) \le C \quad \textit{for } C \textit{ depending only on } K \textit{ and } \|\gamma\|_{W^{1,1}(\mathbb{T})}. \tag{4.8}$$

*Remark* 3. Note that $\tilde{d}_0(\xi)$ is just the winding number of the curve $\gamma$ about the point $\xi$.

*Remark* 4. Observe also that

$$\int_{\mathbb{R}^2} \tilde{d}_0(\xi)\mathcal{L}^2(d\xi) = \frac{1}{2}\int_0^{2\pi} \gamma(\theta) \wedge \frac{d\gamma}{d\theta}\, d\theta =: A. \tag{4.9}$$

This number is just the area (with sign and multiplicity) enclosed by the curve $\gamma$. This is proved by a well-known calculation that we recall below. It is a classical fact (see [9] for example) that for a homogeneous function as above, the distributional determinant of $D^2u$ (which is a distribution on $\mathbb{R}^2$ rather than a measure in the product space) is exactly $A\delta_0$, which by (4.9) is just the marginal on the horizontal $\mathbb{R}^2$ of the measure Det $D^2u$ in the product space. To prove (4.9), fix $R$ such that the image of $\gamma$ is contained in $B_R$, so that $d_0(\xi) = 0$ for $|\xi| > R$. Then

$$\begin{aligned}\int_{\mathbb{R}^2} \tilde{d}_0(\xi)\mathcal{L}^2(d\xi) &= \frac{1}{2\pi}\int_{B_R}\int_0^{2\pi} \frac{\gamma-\xi}{|\gamma-\xi|} \wedge \frac{d}{d\theta}\left(\frac{\gamma-\xi}{|\gamma-\xi|}\right) d\theta\, \mathcal{L}^2(d\xi)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{B_R}\frac{\gamma-\xi}{|\gamma-\xi|^2}\mathcal{L}^2(d\xi)\right) \wedge \frac{d}{d\theta}(\gamma-\xi)\, d\theta\end{aligned}$$

However, it is a classical fact, due to Newton, that $\int_{B_R} \frac{\gamma-\xi}{|\gamma-\xi|^2} d\xi = \gamma$ when $R > |\gamma|$, so (4.9) follows.

We first assume Lemma 4.1 and use it to give the

*proof of Proposition 4.1.* Since $u$ is homogeneous, we can write $u(re^{i\theta}) = f(\theta)$ for some $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

We first prove the theorem under the assumption that $f$ is smooth. Then it follows directly from Lemma 4.1 that for $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$,

$$\int \phi(x,\xi)\,\mathrm{Det}\,D^2u(dx,d\xi) \;=\; [du](\phi d\xi^1 \wedge d\xi^2) \;=\; \int_{\mathbb{R}^2} \phi(0,\xi)\tilde{d}_0(\xi)d\xi. \tag{4.10}$$

So to prove the proposition for smooth $f$, it suffices to show that

$$\tilde{d}_0(\xi) = 1 - \frac{1}{2}\#[\text{sign changes of } (u(x) - \xi\cdot x)|_{S^1}] \tag{4.11}$$

for $\xi \notin$ Image($\gamma$). The smoothness assumption will be dropped at the last step of the proof. We will write $d_0(\xi)$ to denote the right-hand side of (4.11).

We will occasionally write $\tilde{d}_0(\xi; u)$ or $d_0(\xi; u)$ when we need to indicate the dependence of $\tilde{d}_0$ or $d_0$ on $u$.

**1**. Initially for simplicity we assume that $0 \notin \text{Image}(\gamma)$, and we claim that $\tilde{d}_0(0) = d_0(0)$. Let us write $\nu(\theta) = (\cos\theta, \sin\theta)$ and $\tau(\theta) = \nu'(\theta)$. Then easy computations show that

$$\gamma(\theta) = Du(re^{i\theta}) = \nu f(\theta) + \tau f'(\theta).$$

Also, $\gamma'(\theta) = \tau(f + f'') + \nu f' + \tau' f' = \tau(f + f'')$, since $\tau' = -\nu$. Thus

$$2\pi\tilde{d}_0(0) = \int_{\mathbb{T}} \frac{\gamma \times \gamma'}{|\gamma|^2} = \int_{\mathbb{T}} \frac{f^2 + ff''}{f^2 + f'^2} = \int_{\mathbb{T}} \left(1 + \frac{ff'' - f'^2}{f^2 + f'^2}\right).$$

The assumption that $0 \notin \text{Image}(\gamma)$ implies that that $f^2 + f'^2$ never vanishes, so that $f' \neq 0$ whenever $f = 0$. Then $f$ vanishes at a finite collection of points $\theta_1 < \ldots < \theta_{2k} < \theta_{2k+1} = \theta_1 + 2\pi$. Let $I_j$ denote the open interval $(\theta_j, \theta_{j+1})$. Note that $f$ must change sign at every $\theta_j$. (This is why the number of intervals must be even.) Then we have

$$\int_{\mathbb{T}} \frac{ff'' - f'^2}{f^2 + f'^2} = \sum_{j=1}^{2k} \int_{\theta_j}^{\theta_{j+1}} \frac{f^2}{f^2 + f'^2} \frac{ff'' - f'^2}{f^2} d\theta.$$

On $I_j$, we make the change of variables $u = f'/f$, so that $du = \frac{(f''f - f'^2)}{f^2} d\theta$. Note that $\lim_{\theta \nearrow \theta_{j+1}} f'(\theta)/f(\theta) = -\infty$ and $\lim_{\theta \searrow \theta_j} f'(\theta)/f(\theta) = +\infty$ for every $j$, so that

$$\int_{\theta_j}^{\theta_{j+1}} \frac{f^2}{f^2 + f'^2} \frac{ff'' - f'^2}{f^2} \, d\theta = \int_{\infty}^{-\infty} \frac{1}{1 + u^2} du = \arctan(u)|_{\infty}^{-\infty} = -\pi$$

for every $j$. Combining these, we find that

$$\tilde{d}_0(0) = 1 - \frac{1}{2}\#(\text{sign changes of } f) = 1 - \frac{1}{2}\#(\text{sign changes of } u|_{S^1}) = d_0(0).$$

**2**. For general $\xi \in \mathbb{R}^2$ such that $\xi \notin \text{Image}(\gamma)$, define $v(x) = u(x) - \xi \cdot x$, and note that $Dv(re^{i\theta}) = \gamma(\theta) - \xi$. Then

$$d_0(\xi; u) = d_0(0; v) = \tilde{d}_0(0; v) = \tilde{d}_0(\xi; u).$$

In view of (4.10), we have proved the proposition if $f$ is smooth.

**3**. Finally, assume only that $u$ is homogeneous and a Monge-Ampère function. As remarked elsewhere, the fact that $u$ is a Monge-Ampère function implies that $Du$ has bounded variation, and hence that $f'$ has bounded variation. Let $f_k$ be a sequence of smooth functions $\mathbb{T} \to \mathbb{R}$ such that $\|f_k\|_{W^{2,1}} \leq C$ and such that $f_k \to f$ uniformly. It then follows from (4.8) that for every compact $K \subset \mathbb{R}^2$, there exists a constant $C$ such that $\|[du_k]\|(K \times \mathbb{R}^2) \leq C$. Hence Corollary 1.1 implies that $\operatorname{Det} D^2 u_k \rightharpoonup \operatorname{Det} D^2 u$ weakly as measures.

The uniform convergence $f_k \to f$ implies that $d_0(\xi; u_k) \to d_0(\xi, u)$ for a.e. $\xi$. Also, it follows from (4.16), established in the proof of Lemma 4.1 below, that $\|Dd_0(\cdot; u_k)\|_{L^1} = \|D\tilde{d}_0(\cdot; u_k)\|_{L^1} \le C\|f_k\|_{W^{2,1}}$, so that in fact $d_0(\cdot; u_k) \to d_0(\cdot; u)$ in $L^1$. Thus

$$\begin{aligned}\int \phi(x,\xi)\,\mathrm{Det}\,D^2u(dx,d\xi) &= \lim_k \int \phi(x,\xi)\,\mathrm{Det}\,D^2u_k(dx,d\xi)\\ &= \lim_k \int \phi(0,\xi)d_0(\xi;u_k)\mathcal{L}^2(d\xi)\\ &= \int \phi(0,\xi)d_0(\xi;u)\mathcal{L}^2(d\xi).\end{aligned}$$

□

Next we present the

*proof of Lemma 4.1.* It is convenient to write

$$\begin{aligned}G_{Du}(\psi) &= \int_{\mathbb{R}^2} [\psi_0(x, Du(x))dx^1 \wedge dx^2 + \psi_1^{ij}(x, Du)dx^i \wedge d(u_{x_j})],\\ T(\psi) &= \int_{\mathbb{R}^2} \psi_2(0,\xi)\tilde{d}_0(\xi)\mathcal{L}^2(d\xi)\end{aligned}$$

for $\psi$ of the form (4.5). Then we must show that

$$[du] = G_{Du} + T \tag{4.12}$$

$G_{Du}$ is just the current associated with integration over the graph of $Du$. In general one also expects a term of the form $\int \psi_2(x, Du)d(u_{x_1}) \wedge d(u_{x_2})$, but in this case it vanishes, since $d(u_{x_1}) \wedge d(u_{x_2}) = \det D^2u\; dx^1 \wedge dx^2 = 0$, due to the homogeneity of $u$.

To verify (4.12), we must check that $G_{Du} + T$ is an integral current and satisfies the defining properties (1.3) - (1.6). The last condition (1.6) follows directly from the definitions. Condition (1.5) states that $[du]$ has finite mass in $K \times \mathbb{R}^2$ for any compact subset $K$ of $\mathbb{R}^2$. The mass of $G_{Du}$ in $K \times \mathbb{R}^2$ is bounded by

$$C\int_K (1 + |D^2u| + |\det D^2u|) = C\int_K (1 + |D^2u|) \tag{4.13}$$

and hence is finite if $K$ is compact. And the mass of $T$ is just the $L^1$ norm of $\tilde{d}_0$. From the definition one sees rather easily that $\tilde{d}_0(\xi) = 0$ if $|\xi| \ge \|\gamma\|_{L^\infty}$. Writing $R = \|\gamma\|_{L^\infty}$, it follows that

$$\|\tilde{d}_0\|_{L^1} \le \frac{1}{2\pi}\int_{|\xi|\le R}\int_0^{2\pi} \frac{|\gamma'(\theta)|}{|\gamma(\theta) - \xi|}d\theta\mathcal{L}^2(d\xi) \le CR\int_0^{2\pi} |\gamma'|d\theta.$$

Since $R = \|\gamma\|_{L^\infty} \le C\|\gamma\|_{W^{1,1}}$, we conclude that $\|\tilde{d}_0\|_{L^1} \le C\|\gamma\|^2_{W^{1,1}}$. Thus (1.5) holds, and (4.8) also follows, since the right-hand side of (4.13) can clearly be estimated in term of $\|\gamma\|_{W^{1,1}}$ and $K$.

For the Lagrangian condition (1.4) we must check that if $\phi$ is any smooth compactly supported function, then $(G_{Du} + T)(\phi\omega) = 0$, where $\omega = dx^1 \wedge d\xi^1 + dx^2 \wedge d\xi^2$. The definition easily implies that $G_{Du}(\phi\omega) = 0$ — this is just the fact that $D^2u$ is symmetric. And it is clear from the definition of $T$ that $T(\phi\omega) = 0$.

It is rather clear that both $G_{Du}$ and $T$ are integer multiplicity rectifiable, and hence that $G_{Du} + T$ has the same property.

Finally, note that (1.3) is equivalent to

$$\partial T = -\partial G_{Du}. \tag{4.14}$$

To check this, recall first that if we write $\eta = \eta_0^i(x,\xi)dx^i + \eta_1^j(x,\xi)d\xi^j = \eta_0 + \eta_1$, then

$$\partial G_{Du}(\eta) = \int_0^{2\pi} \eta_1^j(0,\gamma(\theta))\frac{d}{d\theta}\gamma^j(\theta)d\mathcal{H}^1(\theta).$$

This is well-known; for example, it is a special case of results in [9] section 3.2.2. For the sake of completeness, however, we give a proof in Lemma 4.2 below. Note also that, in view of the definition of $T$, for $\eta$ as above we have

$$\partial T(\eta) = T(d\eta) = T(\eta^j_{1,\xi_k} d\xi^k \wedge d\xi^j) = \int_{\{0\}\times\mathbb{R}^2} (\eta^2_{1,\xi_1} - \eta^1_{1,\xi_2})\tilde{d}_0(\xi)d\mathcal{H}^2. \tag{4.15}$$

We write $\eta^j(\xi) = \eta_1^j(0,\xi)$, and we identify $\eta$ with the vector field $(\eta^1,\eta^2)$. We must now check that

$$\int_0^{2\pi} \eta(\gamma(\theta))\cdot\gamma'(\theta)d\theta = \int_{\mathbb{R}^2} (\eta^2_{\xi_1} - \eta^1_{\xi_2})\tilde{d}_0(\xi)d\xi, \qquad \eta \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2). \tag{4.16}$$

Fix $\eta \in C_c^\infty$ and let $f(\xi) = -D^\perp \cdot \eta$, where $D^\perp = (-\partial_{\xi_2}, \partial_{\xi_1})$. We use the notation $\Delta^{-1} f = \frac{1}{2\pi}\ln|\cdot| * f$, and we write

$$\zeta(\xi) = D^\perp\Delta^{-1}f(\xi) = \int_{\mathbb{R}^2} \frac{1}{2\pi}\frac{(\xi-a)^\perp}{|\xi-a|^2}f(a)da.$$

Then $D^\perp\cdot\zeta = D^\perp\cdot D^\perp\Delta^{-1}f = -f = D^\perp\cdot\eta$. Thus $\eta$ and $\zeta$ differ by a gradient, which implies that

$$\int_0^{2\pi} \eta(\gamma(\theta))\cdot\gamma'(\theta)d\theta = \int_0^{2\pi} \zeta(\gamma(\theta))\cdot\gamma'(\theta)d\theta.$$

On the other hand, expanding $\zeta$ yields

$$\int_0^{2\pi} \zeta(\gamma(\theta)) \cdot \gamma'(\theta) d\theta = \int_0^{2\pi} \int_{\mathbb{R}^2} f(a) \frac{1}{2\pi} \frac{(\gamma - a)^\perp}{|\gamma - a|^2} \cdot \gamma'(\theta) da \; d\theta$$
$$= \int_{\mathbb{R}^2} f(a) \tilde{d}_0(a) da$$

proving (4.16). □

We conclude this discussion by giving a proof of the standard lemma used above.

**Lemma 4.2.** *Suppose that $v : \mathbb{R}^2 \to \mathbb{R}^2$ is homogeneous of degree* 0*, and smooth away from the origin. Let us write $v(re^{i\theta}) = \gamma(\theta)$, for $\gamma : \mathbb{R} \to \mathbb{R}^2$ smooth and $2\pi$ periodic. Let $V(x) = (x, v(x))$, and let $G_v = V_\#([\mathbb{R}^n \setminus \{0\}])$, so that if $\phi$ has the form* (1.1) *(with $n = 2$) then*

$$G_v(\phi) = \int_{\mathbb{R}^2 \setminus \{0\}} V^\# \phi = \int_{\mathbb{R}^2 \setminus \{0\}} \sum_{|\alpha| + |\beta| = 2} \phi^{\alpha\beta}(x, v(x)) \sigma(\alpha, \bar{\alpha}) M_{\bar{\alpha}}^\beta(Dv).$$

*Then for $\eta = \eta_0^i(x,\xi) dx^i + \eta_1^j(x,\xi) d\xi^j = \eta_0 + \eta_1$,*

$$\partial G_v(\eta) = \int_0^{2\pi} \eta_1^j(0, \gamma(\theta)) \frac{d}{d\theta} \gamma^j(\theta) d\mathcal{H}^1(\theta).$$

*Proof.* Let $\chi : [0, \infty) \to [0, 1]$ be a smooth nondecreasing function such that $\chi(r) = 0$ for $r \leq 1$ and $\chi(r) = 1$ for $r \geq 2$. Let $\chi_s(r) = \chi(r/s)$, and let $\zeta_s(x) = \chi_s(|x|)$.

Then

$$\partial G_v(\eta) = G_v(d\eta) = \lim_{r \to 0} [G_v(\zeta_s d\eta) + G_v((1 - \zeta_s) d\eta)].$$

We first claim that $\lim_{s \to 0} G_v((1 - \zeta_s) d\eta) = 0$. Indeed, $G_v((1 - \zeta_s) d\eta)$ has the form

$$G_v((1 - \zeta_s) d\eta) = \int_{|x| \leq 2s} (1 - \zeta_s) [\phi_0(x, v) + \phi_1^{ij}(x, v) v_{x_j}^i + \phi_2(x, v) \det Dv] dx.$$

for certain functions made out of derivatives of the components of $\eta$. The homogeneity of $v$ implies that $0 = x_i v_{x_i}^j$, and this implies that $\det Dv = 0$. It further implies that $|v_{x_j}^i(x)| \leq C|x|^{-1}$ whence

$$|G_v((1 - \zeta_s) d\eta)| \; \leq C \int_{|x| \leq 2s} [1 + |x|^{-1}] dx \longrightarrow 0 \qquad \text{as } s \to 0.$$

Next,

$$G_v(\zeta_s d\eta) = G_v(d(\zeta_s \eta)) - G_v(d\zeta_s \wedge \eta) = \partial G_v(\zeta_s \eta) - G_v(d\zeta_s \wedge \eta).$$

Note that

$$\partial G_v(\zeta_s \eta) = \partial V_\#([\mathbb{R}^2 \setminus \{0\}])(\zeta_s \eta) = V_\# \partial([\mathbb{R}^2 \setminus \{0\}])(\zeta_s \eta) = 0.$$

The exchange of $\partial$ and $V_\#$ is justified because $\zeta_s \eta$ is supported away from the origin (the only point where $V$ fails to be smooth), and the last equality is geometrically clear. Alternately, the above identity can be checked via an integration by parts. Either way we conclude that

$$\partial G_v(\eta) = \lim_{r \to 0} G_v(d\zeta_s \wedge \eta).$$

The expression on the right-hand side can be expanded as

$$G_v(d\zeta_s \wedge \eta) = \int_{s \le |x| \le 2s} \chi_s'(|x|) d|x| \wedge \left[ \eta_0^i(x,v) dx^i + \eta_1^j(x,v) dv^j \right].$$

As above, simple scaling considerations show that the terms involving $\eta_0$ vanish as $s \to 0$. And the homogeneity implies that $dv^j = \frac{d}{d\theta}\gamma^j(\theta) d\theta$, so that rewriting the remaining terms in polar coordinates (and writing $dr \wedge d\theta$ as $dr d\theta$, as is standard) yields

$$\partial G_v(\eta) = \lim_{s \to 0} G_v(d\zeta_s \wedge \eta_1) = \lim_{s \to 0} \int_{s \le |x| \le 2s} \chi_s'(r) \eta_1^j(re^{i\theta}, \gamma(\theta)) \frac{d}{d\theta}\gamma^j(\theta) dr d\theta.$$

The lemma follows from this identity by integrating first in the radial variable and using the elementary fact that

$$\lim_{s \to 0} \int_0^\infty \chi_s'(r) f(r) dr = \lim_{s \to 0} \int_s^{2s} \chi_s'(r) f(r) dr \;=\; f(0).$$

□

## 5. OPEN PROBLEMS

5.1. **regularity.** Investigate the continuity properties of Monge-Ampère functions. One might hope, based on the examples presented above, that a Monge-Ampère function in $\Omega \subset \mathbb{R}^n$ is $C^{0,2/n+1}$. Note that this is true when $n = 1$.

Lemma 3.2 implies that no proof relying only on control of $\operatorname{Det} D^2 u$ can control the $C^{0,\alpha}$ seminorm for $\alpha > 1/n$, so that if the exponent $\alpha = 2/(n+1)$ is correct, it will require a more subtle proof.

We remark that, as far as we know, it has not even been shown that Monge-Ampère functions are continuous.

5.2. **local representations of Det $D^2u$.** As far as I know, every result along the lines of (4.1), (4.2), giving a local representation of $\operatorname{Det} D^2 u$, is proved under some *a priori* regularity assumptions, such as those in (4.3) (or analogous assumptions in the context of geometric sets and normal cycles, the setting in which most results I know of are stated and proved).

Can one prove that (4.1), (4.2) hold for arbitrary Monge-Ampère functions, without any additional regularity assumptions? One difficulty is that, as emphasized above, regularity properties of Monge-Ampère functions are not well understood, so that it is not clear whether one can for example expect the right-hand side of (4.2) to be well-defined $\mathcal{H}^0(dx) \times \mathcal{L}^n(d\xi)$ almost everywhere, for general $u \in MA_{loc}(\Omega)$.

5.3. **density.** There are a number of natural density questions which are more or less completely open. For example

Are smooth functions weakly dense in the space of Monge-Ampère functions? For example, given $u \in MA_{loc}(\Omega)$, does there exist a sequence $\{u_k\} \subset C^\infty(\Omega)$ such that $u_k \to u$ in $L^1$ and

$$\sup_k \int_K \sum M^{\alpha\beta}(D^2 u_k)\ dx \leq C(K) < \infty \tag{5.1}$$

for every compact $K \subset \Omega$? This question was raised by Fu [4].

A related question deals with *strong* density of smooth functions. A function $u \in MA(\Omega)$ is said to be strongly approximable by smooth functions if there exists a sequence $\{u_k\}$ of smooth functions on $\Omega$ such that

$$u_k \to u \text{ in } L^1 \text{ and } \quad \mathbf{M}([du_k]) \to \mathbf{M}([du]) \qquad \text{as } k \to \infty$$

If $u \in MA(\Omega)$ is weakly approximable by smooth functions, is it also *strongly* approximable?

Very similar questions have been studied in depth for Cartesian currents. In particular, Giaquinta, Modica, and Souček present counterexamples to approximability of Cartesian currents in [8].

We expect that Proposition 4.1 can be used to show that any Monge-Ampère function on the plane that is homogeneous of degree 1 is strongly approximable by smooth functions. This would still be very far from any general density result, but these considerations are at least mildly encouraging, since in particular it suggests that the counterexamples from [8] have no obvious analog in the setting of Monge-Ampère functions.

5.4. **a variational problem arising in statistics.** Suppose that $\Omega$ is a bounded, open subset of $\mathbb{R}^n$, and that we are given points $x_1, \ldots, x_M \in \Omega$ and numbers $y_1, \ldots, y_M \in \mathbb{R}$. Does the functional

$$I[u] := \sum_{i=1}^{M} |u(x_i) - y_i| + \int |\det D^2 u(x)| dx \tag{5.2}$$

admit a minimizer in the space of functions that vanish on $\partial\Omega$?

When $n = 1$, this functional and variants were studied by Koenker, Ng, and Portnoy in [15], in the context of nonparametric regression. These authors prove that the infimum of this functional is attained in $\{u \in W^{1,1}(\mathbb{R}) : u' \text{ has bounded variation}\}$, where $\int |u''|$ is interpreted as the total variation of $u'$. There is in general no uniqueness, but it is shown that one can find minimizers that are piecewise linear, with any discontinuities in the slope only occuring at the some subset of the points $\{x_i\}$. As a result, they show, one can compute minimizers very efficiently via linear programming.

The functional (5.2) was discussed [17] by Koenker, Mizera, and Portnoy as one of a number of possible higher-dimensional extensions of the one-dimensional version. They rapidly concluded that this functional has a number of pathological features that make it of questionable value for applications in statistics. These same features, however, make it an interesting model problem in the calculus of variations.

The space of Monge-Ampère functions, or something rather like it, might provide a suitable functional setting for studying this problem. This was one of the author's motivations for studying spaces of functions without any sort of convexity conditions in which $\operatorname{Det} D^2 u$ makes sense as a measure.

5.5. **degenerate Monge-Ampère equations.** One of the mains results of [12] is that, if $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a Monge-Ampère function such that $\operatorname{Det} D^2 u = 0$ in the sense that

$$[du](\phi d\xi^1 \wedge d\xi^2) = 0 \qquad \text{for all } \phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$$

then for every $x \in \Omega$, at least one of the following must hold:

**1**. $u$ is affine in an open neighborhood of $x$; or

**2**. There exists a line segment $\ell_x$, passing through $x$ and meeting $\partial\Omega$ at both endpoints, along which $Du$ is constant in the sense that

- every point along $\ell_x$ is a Lebesgue point of $Du$, with the same Lebesgue value; or
- every point on $\ell_x$ belongs to the jump set $J_{Du}$ of $Du$, with same approximate limits on both sides of $\ell_x$.

In particular, *every* point in $\Omega$ is either a Lebesgue point of $Du$ or belongs to the jump set of $Du$.

Similar results have been proved for smoother functions that satisfy the equation $\det D^2 u \equiv 0$, including for example by Hartman and Nirenberg [10] for $u \in C^2$, Kirchheim [14] for $u \in W^{2,\infty}$ and Pakzad [16] for $u \in W^{2,2}$.

What analogous result can be established in dimensions $n \geq 3$?

5.6. **miscellaneous other problems.**

**1**. The main result of Fu [5] is that, if $u$ is a *locally Lipschitz* Monge-Ampère function, then

$$[du] \llcorner (\sum \xi_i dx^i) - [du] \llcorner (p^\# du) = 0 \qquad \text{where } p(x,\xi) := x. \tag{5.3}$$

This rather opaque identity is then used to show for example that if $w$ is $C^{1,1}$ and $u$ is Monge-Ampère and locally Lipschitz, then $w \circ u$ is Monge-Ampère, and a suitable chain rule holds. The proof of (5.3) in [5] relies on some delicate approximation arguments that do not automatically extend to the case of Monge-Ampère functions that are not locally Lipschitz. Thus, one might ask whether this or its corollaries are valid for general Monge-Ampère functions.

We remark that (5.3) is trivially true if $u$ is smooth, and follows by approximation for any Monge-Ampère function that can be weakly approximated by smooth functions. So these questions could be answered by proving suitable density results.

**2**. Is the space of Monge-Ampère functions a vector space? This question was raised by Fu [4], who proves that if $u, v$ are Monge-Ampère functions on $\mathbb{R}^n$, then for $\mathcal{L}^n$ a.e. $y \in \mathbb{R}^n$, $x \mapsto u(x) + v(x-y)$ is Monge-Ampère.

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Department of Mathematics, University of Toronto, Toronto, CA M5S 3G3

*E-mail address*: rjerrard@math.toronto.edu