A Theoretical Basis for Two Neutrinos (*).

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Summary. — It is shown how an antisymmetric tensor of rank two can be split up in a covariant manner to give rise to two 2-component Weyl's equations for a neutrino. The photon may then be regarded as a combination of two such neutrinos.

1. – Introduction.

Recent experimental evidence (1) in support of two kinds of neutrino seems to have revived interest (2) in the neutrino theory of light proposed long ago by DE BROGLIE (3) and JORDAN (4). Both these authors tried to explain the photon as a combination of two neutrinos of essentially the same kind. PRYCE (5) and TOUSCHEK *et al.* (6) have shown that in Jordan's theory the photon must be longitudinally polarized.

The object of this paper is to show how an antisymmetric tensor of rank 2 can be split up in a covariant manner to give rise to two « different » 2-component Weyl's equations. The photon can then be thought of as a combination of two such neutrinos.

(*) *Note added in proof.* - An analysis simular to the present work has also been proposed by S. K. BOSE and N. BISWAS at the Tata Institute for Fundamental Research. I am indebted to Dr. BOSE for sending me a preprint of his work.

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(2) S. A. BLUDMAN: *Nuovo Cimento*, **27**, 751 (1963).

(3) L. DE BROGLIE: *Compt. Rend.*, **195**, 536 (1932); **197**, 1377 (1932).

(4) P. JORDAN: *Zeits. Phys.*, **93**, 464 (1935).

(5) M. H. L. PRYCE: *Proc. Roy. Soc.*, A **165**, 247 (1938).

(6) I. M. BARBOUR, A. BIETTI and B. F. TOUSCHEK: *Nuovo Cimento*, **28**, 452 (1963).

2. – Two neutrinos.

Consider an antisymmetric tensor $f_{\mu\lambda}$ in Minkowski space-time. We shall use co-ordinates x_μ $(=x, y, z, ict)$, $\mu=1, 2, 3, 4$, so that the metric tensor is $\delta_{\mu\lambda}$, and consider only proper Lorentz transformations

(1)
$$\begin{cases} x_\mu \to x'_\mu = \alpha_{\mu\nu} x_\nu \,, \\ \det \alpha = +1 \,. \end{cases}$$

Under (1) $f_{\mu\lambda}$ transforms as

(2)
$$f_{\mu\lambda} \to f'_{\mu\lambda} = \alpha_{\mu\alpha}\alpha_{\lambda\beta} f_{\alpha\beta} \,.$$

Consider the dual of $f_{\mu\lambda}$,

(3)
$$f^*_{\mu\lambda} = \tfrac{1}{2}\varepsilon_{\mu\lambda\alpha\beta} f_{\alpha\beta} \,,$$

where $\varepsilon_{\mu\lambda\alpha\beta}$ is the completely antisymmetric Levi-Civita symbol. Then one can write

(4)
$$f_{\mu\lambda} = \tfrac{1}{2}(f_{\mu\lambda} + f^*_{\mu\lambda}) - \tfrac{1}{2}(f_{\mu\lambda} - f^*_{\mu\lambda}) \,.$$

The first term is self-dual and the second anti-self-dual. The above decomposition is an invariant process. That is, the self-dual (anti-self-dual) part is transformed again into a self-dual (anti-self-dual) part under (1). This is true only for proper Lorentz transformations.

Consider now a self-dual antisymmetric tensor $f^s_{\mu\lambda}$, *i.e.* for which

(5)
$$f^s_{\mu\lambda} = f^{s*}_{\mu\lambda} \,.$$

It has only three independent components, which we denote by a, b, c. Written out in full (5) gives according to (3)

(6)
$$\begin{cases} a = f^s_{12} = f^s_{34} \,, \\ b = f^s_{23} = f^s_{14} \,, \\ c = f^s_{31} = f^s_{24} \,. \end{cases}$$

It is useful to see how the three independent components transform under (1). It is clear that $\frac{1}{4} f^s_{\mu\lambda} f^{s*}_{\mu\lambda} = a^2 + b^2 + c^2$ is an invariant. a, b, c transform therefore orthogonally, *i.e.*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}' = \begin{pmatrix} A_{11} & \cdot & \cdot & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & \cdot & \cdot & A_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} ,$$

where A_{ik} $(i, k = 1, 2, 3)$ is an orthogonal matrix. One can compute $A_{ik}(\alpha_{\mu\nu})$ using (2) and (6):

$$(7) \qquad A(\alpha) = \begin{pmatrix} \alpha_{11}\alpha_{22} + \alpha_{24}\alpha_{13} - \alpha_{21}\alpha_{12} - \alpha_{23}\alpha_{14} & \begin{matrix} \alpha_{23}\alpha_{12} + \alpha_{11}\alpha_{24} - \alpha_{22}\alpha_{13} - \alpha_{21}\alpha_{14} \\ \alpha_{24}\alpha_{12} + \alpha_{21}\alpha_{13} - \alpha_{22}\alpha_{14} - \alpha_{23}\alpha_{11} \end{matrix} \\ \alpha_{13}\alpha_{44} + \alpha_{11}\alpha_{42} - \alpha_{43}\alpha_{14} - \alpha_{41}\alpha_{12} & \begin{matrix} \alpha_{11}\alpha_{44} + \alpha_{12}\alpha_{43} - \alpha_{13}\alpha_{42} - \alpha_{14}\alpha_{41} \\ \alpha_{12}\alpha_{44} + \alpha_{13}\alpha_{41} - \alpha_{14}\alpha_{42} - \alpha_{43}\alpha_{11} \end{matrix} \\ \alpha_{31}\alpha_{12} + \alpha_{33}\alpha_{14} - \alpha_{11}\alpha_{32} - \alpha_{34}\alpha_{13} & \begin{matrix} \alpha_{31}\alpha_{14} + \alpha_{32}\alpha_{13} - \alpha_{33}\alpha_{12} - \alpha_{34}\alpha_{11} \\ \alpha_{32}\alpha_{14} + \alpha_{33}\alpha_{11} - \alpha_{34}\alpha_{12} - \alpha_{31}\alpha_{13} \end{matrix} \end{pmatrix}$$

For a special Lorentz transformation given by

$$(8) \qquad \alpha = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

$$\beta = \frac{v}{c}, \qquad \gamma = (1-\beta^2)^{-\frac{1}{2}}.$$

$$(9) \qquad A(\alpha) = \begin{pmatrix} \gamma & 0 & -i\beta\gamma \\ 0 & 1 & 0 \\ i\beta\gamma & 0 & \gamma \end{pmatrix}.$$

Now a self-dual (or an anti-self-dual) antisymmetric tensor cannot describe an electromagnetic field because in that case there should exist a vector $A_\mu = (\boldsymbol{A}, i\Phi)$ such that $f^s_{\mu\lambda} = \partial_\lambda A_\mu - \partial_\mu A_\lambda$. No such vector with three real components and one imaginary can exist if $f^s_{\mu\lambda}$ were to be self-dual because (6) would then imply that

$$\partial_y A_x - \partial_x A_y = \frac{1}{ic}\,\partial_t A_z - i\partial_z \Phi\,,$$

which violates reality conditions.

Consider however the following 4-component quantity (*)

$$(10) \qquad \psi^s = \begin{pmatrix} \varphi_1 \\ i\varphi_1 \\ -\varphi_2 \\ i\varphi_2 \end{pmatrix},$$

and set $f^s_{\mu\lambda} = \partial_\lambda \psi^s_\mu - \partial_\mu \psi^s_\lambda$.

Since $f^s_{\mu\lambda}$ is self-dual the first equation of (6) gives

$$f^s_{12} = \partial_2\varphi_1 - i\partial_1\varphi_1 = -\partial_4\varphi_2 - i\partial_3\varphi_2 = f^s_{34}\,.$$

Multiplying by i we get

$$(11) \qquad \partial_1\varphi_1 + i\partial_2\varphi_1 - \partial_3\varphi_2 + i\partial_4\varphi_2 = 0\,.$$

Similarly the second equation of (6) gives

$$f^s_{23} = i\partial_3\varphi_1 + \partial_2\varphi_2 = \partial_4\varphi_1 - i\partial_1\varphi_2 = f^s_{14}\,.$$

Multiplying by $-i$ we get

$$(12) \qquad \partial_1\varphi_2 - i\partial_2\varphi_2 + \partial_3\varphi_1 + i\partial_4\varphi_1 = 0\,.$$

The third equation of (6) gives again (12)

$$f^s_{31} = -\partial_1\varphi_2 - \partial_3\varphi_1 = i\partial_4\varphi_1 - i\partial_2\varphi_2 = f^s_{24}\,.$$

or

$$\partial_1\varphi_2 - i\partial_2\varphi_2 + \partial_3\varphi_1 + i\partial_4\varphi_1 = 0\,.$$

Thus the condition (6) for a self-dual antisymmetric tensor together with (10) give two eqs. (12) and (11) for φ_1 and φ_2, which by introducing the Pauli matrices

$$(13) \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(*) See Appendix.

and

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

can be combined into one single equation

$$(\sigma_k \partial_k + i\partial_4)\varphi = 0 \,, \tag{14}$$

which is nothing but Weyl's 2-component neutrino equation.

We thus see that the 2-component neutrino can also be completely described by a self-dual antisymmetric tensor behaving very much like the electromagnetic field tensor.

What about the anti-self-dual case? It is possible to follow a similar procedure in the case of an anti-self-dual antisymmetric tensor for which

$$f^a_{\mu\lambda} = -f^{a*}_{\mu\lambda} \,, \tag{15}$$

i.e.

$$\left|\begin{aligned} f^a_{24} &= f^a_{13} \,, \\ f^a_{34} &= f^a_{21} \,, \\ f^a_{14} &= f^a_{32} \,. \end{aligned}\right. \tag{16}$$

If one considers just the arrangement of the tensor indices it is possible to pass over from the set (16) to (6) by one single interchange of the indices, *e.g.*, one and three. So we consider now the following quantity

$$\psi^a = \begin{pmatrix} -\chi_2 \\ i\chi_1 \\ \chi_1 \\ i\chi_2 \end{pmatrix} , \tag{17}$$

and set this time $f^a_{\mu\lambda} = \partial_\lambda \psi^a_\mu - \partial_\mu \psi^a_\lambda$.

However (16) combined with (17) do not give again exactly the same set of eqs. (11) and (12) because the co-ordinate indices now get interchanged. Thus $f^a_{24} = f^a_{13}$ gives

$$\partial_1 \chi_1 - i\partial_2 \chi_2 + \partial_3 \chi_2 + i\partial_4 \chi_1 = 0 \tag{18}$$

and $f^a_{34} = f_{21}$ gives

$$-\partial_2\chi_2 - i\partial_1\chi_1 + \partial_4\chi_1 - i\partial_3\chi_2 = 0\,,$$

which on multiplication by i gives (18) again. And $f^a_{14} = f^a_{32}$ gives

$$i\partial_3\chi_1 - \partial_2\chi_1 - \partial_4\chi_2 - i\partial_1\chi_2 = 0\,.$$

Multiplying by $-i$ we get

(19) $$-\partial_1\chi_2 + i\partial_2\chi_1 + \partial_3\chi_1 + i\partial_4\chi_2 = 0\,.$$

Introducing

(20) $$\sigma'_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma'_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma'_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

(18) and (19) can be combined into

(21) $$(\sigma'_k\partial_k + i\partial_4)\chi = 0\,,$$

which differs from (14) only in the respect that the set σ'_k differs from σ_k only by the interchange of the indices one and three. As a result we have for the σ'_k's

(22) $$\left|\begin{array}{l} \sigma'_1\sigma'_2 = -i\sigma'_3\,, \\ \sigma'_2\sigma'_3 = -i\sigma'_1\,, \\ \sigma'_3\sigma'_1 = -i\sigma'_2\,. \end{array}\right.$$

We note a few algebraic properties of ψ^s and ψ^a. We have

$$\psi^s = B\psi^s\,,$$

where

$$B = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

and

$$B = X\gamma_5\,,$$

where

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

Also

$$\psi^a = C\psi^a\,,$$

where

$$C = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

3. – Photon.

Consider now

$$\psi = \psi^s + \psi^a = \begin{pmatrix} \varphi_1 - \chi_2 \\ i(\varphi_1 + \chi_1) \\ \chi_1 - \varphi_2 \\ i(\varphi_2 + \chi_2) \end{pmatrix}; \tag{23}$$

and let $f_{\mu\lambda} = \partial_\lambda \psi_\mu - \partial_\mu \psi_\lambda = f^s_{\mu\lambda} + f^a_{\mu\lambda}$.

Then the second set of Maxwell's equations

$$\partial_\lambda f_{\mu\lambda} = 0 \tag{24}$$

follows automatically in view of (14) and (21). It will suffice to show this just for the case $\mu = 1$. We have from (25)

$$\begin{aligned} f_{12} &= \partial_2(\varphi_1 - \chi_2) - i\partial_1(\varphi_1 + \chi_1)\,, \\ f_{13} &= \partial_3(\varphi_1 - \chi_2) - \partial_1(\chi_1 - \varphi_2)\,, \\ f_{14} &= \partial_4(\varphi_1 - \chi_2) - i\partial_1(\varphi_2 + \chi_2)\,, \end{aligned}$$

$$\therefore\ \partial_2 f_{12} + \partial_3 f_{13} + \partial_4 f_{14} = (\partial_2^2 + \partial_3^2 + \partial_4^2)(\varphi_1 - \chi_2) - i\partial_2\partial_1(\varphi_1 + \chi_1) - \\ - \partial_3\partial_1(\chi_1 - \varphi_2) - i\partial_4\partial_1(\varphi_2 + \chi_2)\,.$$

Since $\Box\varphi_1 = \Box\chi_2 = 0$, the right hand side becomes

$$\partial_1(-\partial_1\varphi_1 - i\partial_2\chi_1 + \partial_3\varphi_2 - i\partial_4\varphi_2) + \partial_1(\partial_1\chi_2 - i\partial_2\chi_1 - \partial_3\chi_1 - i\partial_4\chi_2) = 0\,,$$

in view of (11) and (19).

The same can be shown for $\mu = 2, 3, 4$.

4. – Conclusion.

We have thus shown that the photon can be considered as a combination of two neutrinos, one described by (14) and the other by (21) and that the neutrinos described by (14) and (21) can be represented by a self-dual and an anti-self-dual antisymmetric tensor, respectively. The two neutrinos will behave identically when they are free but their interactions with other particles may be different. The problem of how they differ in their interactions remains to be investigated.

* * *

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Appendix

We consider here the transformation properties of φ and χ. They must of course transform like spinors if eqs. (14) and (21) are to have an invariant form. It is, however, not immediately clear that they do so from the way they are introduced in (10) and (17).

We must have (ψ being a vector) according to (23)

$$\sum_\mu \psi_\mu^2 \ \text{invariant}\,, \qquad i.e. \qquad \varphi_1\chi_1 + \varphi_2\chi_2 + \varphi_1\chi_2 + \chi_1\varphi_2 \ \text{invariant}\,.$$

This is a singular bilinear form. Written in matrix form we have then

$$\varphi^T \eta \chi \text{ invariant}.$$

where φ^T is the transpose of φ and

$$\eta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

a singular matrix.

Let

$$\varphi' = A\varphi, \qquad \chi' = B\chi.$$

Then we must have

$$(\varphi')^T \eta \chi' = \varphi^T A^T \eta B \chi = \varphi^T \eta \chi.$$

We can assure this by first demanding that η commutes with A^T and then taking $B = (A^T)^{-1}$.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $\eta A^T = A^T \eta$ implies $a = d$ and $b = c$. We thus obtain a class of matrices of the type

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \tag{A.1}$$

If we further impose the condition that

$$\det A = \alpha^2 - \beta^2 = 1,$$

we have

$$B = (A^T)^{-1} = A^{-1} = \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix},$$

with $\det B = 1$ also.

The matrices (A.1), together with the unimodularity condition form an Abelian subgroup of C_2 (the group of complex 2×2 unimodular matrices). The product of two such matrices is again a matrix of the same class and the unit matrix belongs to the class.

We also note that the form of ψ^s in (10) is equivalent to a gauge condition

$$\psi_1^s + i\psi_2^s = 0, \qquad \psi_4^s + i\psi_3^s = 0,$$

which is invariant under the class of gauge transformations: $\psi^a_\mu \to \psi^a_\mu + \partial_\mu f$, where f satisfies the following equations:

$$\partial_1 f + i\partial_2 f = 0 \, , \qquad \partial_4 f + i\partial_3 f = 0 \, .$$

The corresponding gauge condition for ψ^a is

$$\psi^a_3 + i\psi^a_2 = 0 \, , \qquad \psi^a_4 + i\psi^a_1 = 0 \, ,$$

which is invariant under the class of gauge transformations

$$\partial_3 g + i\partial_2 g = 0 \, , \qquad \partial_4 g + i\partial_1 g = 0 \, .$$

The gauge conditions are of course not invariant under the co-ordinate transformations (1), but are invariant under those Lorentz transformations α for which $\alpha B = B\alpha$ and $\alpha C = C\alpha$ respectively. These turn out to be rotations in the (x_1, x_2) and (x_3, x_4) planes for the self-dual case and rotations in the (x_2, x_3) and (x_1, x_4) planes for the anti-self-dual case.

RIASSUNTO (*)

Si mostra come un tensore antisimmetrico di ordine due può essere diviso in modo covariante dando origine per un neutrino a due equazioni di Weyl a due componenti. Il fotone può quindi essere considerato una combinazione di due di questi neutrini.

(*) *Traduzione a cura della Redazione.*