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Quantum Electrodynamics of Confined Nonrelativistic Particles

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We consider a system of finitely many nonrelativistic, quantum mechanical electrons bound to static nuclei. The electrons are minimally coupled to the quantized electromagnetic field; but we impose an ultraviolet cutoff on the electromagnetic vector potential appearing in covariant derivatives, and the interactions between the radiation field and electrons localized very far from the nuclei are turned off. For a class of Hamiltonians we prove exponential localization of bound states, establish the existence of a ground state, and derive sufficient conditions for its uniqueness. Furthermore, we show that excited bound states of the unperturbed system become unstable and turn into resonances when the electrons are coupled to the radiation field. To this end we develop a novel renormalization transformation which acts directly on the space of Hamiltonians. © 1998 Academic Press

Key Words: quantum electrodynamics; nonrelativistic particles; ground state; resonances; Fermi's golden rule; renormalization group.

I. INTRODUCTION: DESCRIPTION OF THE MATHEMATICAL PROBLEM AND MAIN RESULTS

In this paper we establish mathematical results concerning physical phenomena that stood at the origin of quantum theory: that of emission

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and absorption of electromagnetic radiation by systems of nonrelativistic, quantum mechanical matter, such as atoms and molecules.

Our analysis supplies *nonperturbative* results supporting the conventional picture of atoms and molecules interacting with the quantized radiation field provided by low-order quantum mechanical perturbation theory, adding mathematical precision to it. In particular, our results concerning the structure of resonances in the fully interacting theory corresponding to excited energy levels of an atom or molecule decoupled from the radiation field are not entirely part of the conventional wisdom in this field and complement the perturbative analysis, first carried out by Bethe [7, 8], of the Lamb shift.

We conduct our analysis on a model, introduced in Subsection I.2, which is somewhat simpler than the standard model (see Subsection I.1), but which nevertheless retains all the physical features of the latter. In fact, we believe that the infrared problem in our model is more severe than in the standard one. Indeed, as stipulated by the Pauli-Fierz transformation (see Subsection I.1) and will be explained in a subsequent publication, the particular form of the interaction in the standard model (minimal coupling, leading to gauge invariance) can be used to improve the infrared behaviour of the interaction considerably and to make it, at least formally, much more benign than the one considered in this paper. The full standard model will be considered in a subsequent publication. For general background on quantum field theory see [17].

I.1. *The Standard Model of Nonrelativistic Quantum Electrodynamics*

The starting point of our analysis is the following *standard model of nonrelativistic, quantum mechanical matter and radiation.* The system consists of a finite number of nuclei, in the following treated as static, and of electrons treated as nonrelativistic, quantum mechanical point particles coupled to the quantized electromagnetic field. First, we describe the radiation field. Its Hilbert space is given by

$$\mathscr{H}_f \equiv \mathscr{F} := \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3, d^3k) \otimes \mathbb{C}^2)^{\otimes_s n}. \tag{I.1}$$

In this formula, the subscript "f" refers to *photon field.* The space $\mathscr{H}_f = \mathscr{F}$ is the Fock space of photons. The factor $\mathbb{C}^2$ on the r.s. of (I.1) accounts for the two possible polarizations of photons, and $\otimes_s$ denotes a symmetric tensor product appropriate for Bose–Einstein statistics.

The transverse modes of the quantized electromagnetic field are described in terms of the vector potential, $\mathbf{A}$, in the Coulomb gauge: At time $t=0$, $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{x}) := \sum_{\lambda=1,2} \frac{1}{\pi} \int \frac{d^3k}{\sqrt{2\omega(\mathbf{k})}} [\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} a_\lambda^\dagger(\mathbf{k}) + \boldsymbol{\varepsilon}_\lambda(\mathbf{k})^*\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} a_\lambda(\mathbf{k})], \tag{I.2}$$

where $\omega(\mathbf{k}) = |\mathbf{k}|$ is the energy of a photon of momentum $\mathbf{k}$; $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$, $\lambda = 1, 2$, are polarization vectors satisfying

$$\boldsymbol{\varepsilon}_\lambda(\mathbf{k})^* \cdot \boldsymbol{\varepsilon}_\mu(\mathbf{k}) = \delta_{\lambda\mu}, \qquad \mathbf{k} \cdot \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0, \qquad \lambda, \mu = 1, 2;$$

and $a_\lambda^\dagger(\mathbf{k})$, $a_\lambda(\mathbf{k})$ are the usual creation and annihilation operators on $\mathscr{F}$ obeying the canonical commutation relations

$$[a_\lambda^\#(\mathbf{k}_1), a_\mu^\#(\mathbf{k}_2)] = 0, \qquad [a_\lambda(\mathbf{k}_1), a_\mu^\dagger(\mathbf{k}_2)] = \delta_{\lambda\mu}\delta(\mathbf{k}_1 - \mathbf{k}_2), \tag{I.3}$$

where $a^\# = a$ or $a^\dagger$. The vector potential $\mathbf{A}$ and the creation and annihilation operators are unbounded, operator-valued distributions on $\mathscr{F}$. The space $\mathscr{F}$ contains a unique vector Ω, the vacuum vector, with the property that $a_\lambda(\mathbf{k})\Omega = 0$, for all λ and $\mathbf{k}$. The free time-evolution of the radiation field is generated by the Hamiltonian

$$\mathbb{H}_f := \sum_{\lambda=1,2} \int d^3k\, \omega(\mathbf{k})\, a_\lambda^\dagger(\mathbf{k})\, a_\lambda(\mathbf{k}). \tag{I.4}$$

The free time-evolution of the quantized vector potential is given by

$$\mathbf{A}(\mathbf{x}, t) = e^{it\mathbb{H}_f}\, \mathbf{A}(\mathbf{x})\, \mathrm{e}^{-it\mathbb{H}_f},$$

while the evolution of the *free* electric and magnetic fields is given by

$$\mathbf{E}(\mathbf{x}, t) = \frac{\partial}{\partial t} \mathbf{A}(\mathbf{x}, t), \qquad \mathbf{B}(\mathbf{x}, t) = \nabla \wedge \mathbf{A}(\mathbf{x}, t).$$

Here we use units such that $\hbar = c = 1$, where $\hbar$ is Planck's constant and c is the velocity of light.

Now we describe a system consisting of N electrons moving in the electrostatic potential generated by M fixed nuclei of charges $Z_\ell e$ at positions $\mathbf{R}_\ell$, $\ell = 1,\ldots, M$. Its Hamiltonian is given by

$$\mathbb{H}_{el}(R) = \sum_{j=1}^{N} \frac{1}{2m} [\boldsymbol{\sigma}_j \cdot (-i\boldsymbol{\nabla}_j)]^2 + \alpha \mathbb{V}(x, R), \tag{I.5}$$

where m is the mass of an electron, $\boldsymbol{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$, σ_j^x, σ_j^y and σ_j^z are the three Pauli matrices acting on the spin space of the jth electron, $\alpha = e^2/4\pi \cong (137)^{-1}$ is the feinstructure constant, and $\mathbb{V}(x, R) = \sum_{j=1}^N \sum_{\ell=1}^M (-Z_\ell/|\mathbf{x}_j - \mathbf{R}_\ell|) + \sum_{1 \leqslant i < j \leqslant N} (1/|\mathbf{x}_i - \mathbf{x}_j|)$. Here $x = (\mathbf{x}_1, ..., \mathbf{x}_N)$, $R = (\mathbf{R}_1, ..., \mathbf{R}_M)$, $-\sum_{\ell=1}^M Z_\ell \alpha\, |\mathbf{x} - \mathbf{R}_\ell|^{-1}$ is the electrostatic potential at $\mathbf{x}$ corresponding to the charge distribution of the nuclei, and $\alpha/|\mathbf{x}_i - \mathbf{x}_j|$ is the repulsive Coulomb potential between the ith and the jth electron. The subscript "*el*" refers to *electrons*. This operator acts on the Hilbert space

$$\mathscr{H}_{el} := (L^2(\mathbb{R}^3, d^3x) \otimes \mathbb{C}^2)^{\otimes_a N}. \tag{I.6}$$

Moreover, the factor $\mathbb{C}^2$ on the r.h.s. of (I.6) accounts for the spin of electrons, and $\otimes_a$ denotes the anti-symmetric tensor product, in accordance with the Pauli principle.

Now we are ready to describe a sytem composed of N electrons moving in an electrostatic potential $\alpha\mathbb{V}(x, R)$ and interacting with the photon field. It is described by the Hamiltonian

$$\mathbb{H}_{el+f} = \sum_{j=1}^N \frac{1}{2m} [\boldsymbol{\sigma}_j \cdot (-i\nabla_j - e_j \mathbf{A}_\kappa(\mathbf{x}_j))]^2 + \alpha\mathbb{V}(x, R) \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f, \tag{1.7}$$

acting on the Hilbert space of the total system,

$$\mathscr{H} = \mathscr{H}_{el} \otimes \mathscr{H}_f. \tag{I.8}$$

Here $\mathbf{A}_\kappa(\mathbf{x}) = \int d^3y\, \check{\kappa}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{y})$ is the regularized vector potential, where $\check{\kappa}$ is the Fourier transform of a smooth function, κ, with support contained in the ball $\{\mathbf{k}\colon |\mathbf{k}| \leqslant \text{const} \cdot \alpha m\}$, The Hamiltonian $\mathbb{H}_{el+f}$ describes the *standard model* of Quantum Electrodynamics for nonrelativistic particles. The smoothing operation applied to the original vector potential operator $\mathbf{A}(\mathbf{x})$ is called the *ultraviolet cutoff*. It is justified by the fact that the phenomena we study are nonrelativistic in nature as far as particles are concerned. It is needed in order for the Hamiltonian to be well-defined.

A simple change of units exhibits the perturbative character of the present spectral problem. Indeed, we dilate the electron coordinates and photon momenta independently, $(\mathbf{x}_j, \mathbf{k}) \mapsto (\eta \mathbf{x}_j, \mu^{-1}\mathbf{k})$, employing a suitable unitary operator, U_1, on $\mathscr{H}$. Upon making the choice $1/2m\eta^2 = \alpha/\eta = \mu$, we obtain

$$\begin{aligned} \mathbb{H}_1 &:= \mu^{-1} U_1 \mathbb{H}\, U_1^* \\ &= \sum_{j=1}^N [\boldsymbol{\sigma}_j \cdot (-i\nabla_{\mathbf{x}_j} - 2\pi^{1/2}\alpha^{3/2} \mathbf{A}_{\bar{\kappa}}(\alpha \mathbf{x}_j))]^2 \\ &\quad + \mathbb{V}(x, \eta^{-1}R) \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f, \end{aligned} \tag{I.9}$$

where $\bar{\kappa}(\mathbf{k}) := \kappa(\mu\mathbf{k})$. This expression suggests to study the operator $\mu^{-1}U_1 H U_1^*$ as a perturbation of the operator $\mathbb{H}_{el}(\eta^{-1}R) \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f$, because $\alpha \approx 1/137$ is small.

The problem of radiation involves the control of photons at low energies. The feature of the Hamiltonian which determines whether such a control is possible is the small $|\mathbf{k}|$ behaviour of the coupling function $g(\mathbf{k}) := \bar{\kappa}(\mathbf{k})(2\omega(\mathbf{k}))^{-1/2}$, showing how strongly the low energy photons are coupled to the particles. As far as the scaling dimension of the perturbation (see also Section I.3 below) is concerned, $g = \text{const}\ \omega(\mathbf{k})^{-1/2}$, as $|\mathbf{k}| \to 0$, is exactly a borderline case. Some of the results of this paper require that $\int g^2/\omega^2 < \infty$ which the standard model for which $\bar{\kappa}(\mathbf{k}) \to 1$, as $|\mathbf{k}| \to 0$, misses by a wisker. However, on a formal level this problem is removed if we perform a Pauli–Fierz transformation [34].

From now on we write κ for $\bar{\kappa}$. Assuming that the charge distribution of the nuclei is concentrated around the origin in $\mathbb{R}^3$, we unitarily transform the Hamiltonian $\mathbb{H}_1$ in (I.9) by $U_2 := \exp[-i\tau \mathbf{A}_{\bar{\kappa}}(\mathbf{0}) \cdot (\sum_{j=1}^N \mathbf{x}_j)]$. This unitary conjugation leaves all terms in $\mathbb{H}$ unchanged except for

$$U_2(-i\,\nabla_{\mathbf{x}_j})\,U_2^* = -i\,\nabla_{\mathbf{x}_j} + \tau\,\mathbf{A}_{\bar{\kappa}}(\mathbf{0})$$

and

$$U_2 a_\lambda^\dagger(\mathbf{k})\,U_2^* = a_\lambda^\dagger(\mathbf{k}) + \frac{i\tau\kappa(\mathbf{k})}{\pi\sqrt{2\omega(\mathbf{k})}}\left(\sum_{j=1}^N \boldsymbol{\varepsilon}_\lambda(\mathbf{k})^* \cdot \mathbf{x}_j\right).$$

Let $\mathbb{H}_2 := U_2 \mathbb{H}_1 U_2^*$. Choosing $\tau := 2\sqrt{\pi}\,\alpha^{3/2}$ and using that

$$[\boldsymbol{\sigma}\cdot(-i\nabla_{\mathbf{x}} - \tau\mathbf{A}_{\bar{\kappa}}(\alpha\mathbf{x}))]^2 = (-i\nabla_{\mathbf{x}} - \tau\mathbf{A}_{\bar{\kappa}}(\alpha\mathbf{x}))^2 + \tau\alpha(\nabla_{\mathbf{x}} \wedge \mathbf{A}_{\bar{\kappa}})(\alpha\mathbf{x}),$$

we obtain

$$\begin{aligned}\mathbb{H}_2 = {} & \left(\mathbb{H}_{el} + 4\pi\alpha^3 c_\kappa \left(\sum_{j=1}^N \mathbf{x}_j\right)^2\right) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbb{H}_f \\ & + 2\sqrt{\pi}\,\alpha^{3/2}\mathbf{E}_{\bar{\kappa}}(0) \cdot \left(\sum_{j=1}^N \mathbf{x}_j\right) + \pi^{1/2}\alpha^{5/2} \sum_{j=1}^N \{\boldsymbol{\sigma}_j \cdot \mathbf{B}_{\bar{\kappa}}(\alpha\mathbf{x}_j)\} \\ & + \sum_{j=1}^N \{i4\pi^{1/2}\alpha^{3/2}\,\nabla_{\mathbf{x}_j} \cdot [\mathbf{A}_{\bar{\kappa}}(\alpha\mathbf{x}_j) - \mathbf{A}_{\bar{\kappa}}(\mathbf{0})] + 4\pi\alpha^3[\mathbf{A}_{\bar{\kappa}}(\alpha\mathbf{x}_j) - \mathbf{A}_{\bar{\kappa}}(\mathbf{0})]^2\},\end{aligned} \tag{I.10}$$

where $\mathbf{E}_{\bar{\kappa}}(\mathbf{x}) := i[\mathbb{H}_f, \mathbf{A}_{\bar{\kappa}}(\mathbf{x})]$ and $\mathbf{B}_{\bar{\kappa}}(\mathbf{x}) := \nabla \wedge \mathbf{A}_{\bar{\kappa}}(\mathbf{x})$ and

$$\mathbb{H}_{el} := \sum_{j=1}^N -\Delta_j + \mathbb{V}(x, \eta^{-1}R). \tag{I.11}$$

Here, $c_{\bar{\kappa}} = \pi^{-2} \int \kappa(\mathbf{k})^2 \, d^3k$ is a cutoff-dependent constant. This expression shows that the singular behaviour of the coupling function $\bar{\kappa}(\mathbf{k})(2\omega(\mathbf{k}))^{-1/2}$ can be traded for a perturbation growing as $|\sum_{j=1}^{N} \mathbf{x}_j| \to \infty$. The latter does not sound so bad, at least on a conceptual level, since the processes of radiation we consider in this paper involve bound, and therefore exponentially localized, electrons (see Theorem II.1 below). Note that $\mathbb{H}_2$ can be written as

$$\mathbb{H}_2 = \left(\mathbb{H}_{el} + g^2 c_{\bar{\kappa}} \left(\sum_{j=1}^{N} \mathbf{x}_j \right)^2 + g^2 \sum_{j=1}^{N} f(\alpha \mathbf{x}_j) \right) \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f + \mathbb{W}_g, \tag{I.12}$$

$$\mathbb{W}_g = g\,\mathbb{W}_1 + g^2\,\mathbb{W}_2, \tag{I.13}$$

where $g := 2\sqrt{\pi}\,\alpha^{3/2}$, $f(\mathbf{x}) := (4/\pi^2) \int (\bar{\kappa}^2(\mathbf{k}) \sin^2(\mathbf{k} \cdot \mathbf{x}) \, d^3k)/\omega(\mathbf{k})$, and

$$\mathbb{W}_1 = \int \{ \mathbb{G}_{1,0}(\mathbf{k}) \otimes a_\lambda^\dagger(\mathbf{k}) + \mathbb{G}_{0,1}(\mathbf{k}) \otimes a_\lambda^\dagger(\mathbf{k}) \} \, d\mathbf{k}, \tag{I.14}$$

$$\begin{aligned} \mathbb{W}_2 = \int \{ & \mathbb{G}_{2,0}(\mathbf{k}, \mathbf{k}') \otimes a_\lambda^\dagger(\mathbf{k}) \, a_\lambda^\dagger(\mathbf{k}') + \mathbb{G}_{0,2}(\mathbf{k}, \mathbf{k}') \otimes a_\lambda(\mathbf{k}) \, a_\lambda(\mathbf{k}') \\ & + \mathbb{G}_{1,1}(\mathbf{k}, \mathbf{k}') \otimes a_\lambda^\dagger(\mathbf{k}) \, a_\lambda(\mathbf{k}') \} \, dk \, dk'. \end{aligned} \tag{I.15}$$

Here $\mathbb{G}_{m,n}$ are functions of $(\mathbf{k}; \lambda)$ or $(\mathbf{k}, \mathbf{k}'; \lambda, \lambda')$, respectively, with values in the operators on $\mathscr{H}_{el}$. Comparing (I.12)–(I.15) to (I.10), one easily verifies that

$$\begin{aligned} \mathbb{G}_{1,0}(\mathbf{k}, \lambda) &:= \mathbb{G}_{0,1}(\mathbf{k}, \lambda)^* \\ &:= \left(\frac{\bar{\kappa}(\mathbf{k})}{\pi \sqrt{2\omega(\mathbf{k})}} \right) \sum_{j=1}^{N} \Big\{ i\omega(\mathbf{k}) \, \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \mathbf{x}_j + (e^{-i\alpha \mathbf{k} \cdot \mathbf{x}_j} - 1) \\ &\qquad \times \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \nabla_{\mathbf{x}_j} + \frac{i\alpha}{2} e^{-i\alpha \mathbf{k} \cdot \mathbf{x}_j} [\boldsymbol{\sigma}_j \cdot (\mathbf{k} \wedge \boldsymbol{\varepsilon}_\lambda(\mathbf{k}))] \Big\}, \end{aligned} \tag{I.16}$$

$$\begin{aligned} \mathbb{G}_{2,0}(\mathbf{k}, \lambda; \mathbf{k}', \lambda') &:= \mathbb{G}_{0,2}(\mathbf{k}, \lambda; \mathbf{k}', \lambda')^* \\ &:= \left(\frac{\bar{\kappa}(\mathbf{k}) \, \bar{\kappa}(\mathbf{k}')}{2\pi^2 \sqrt{\omega(\mathbf{k}) \, \omega(\mathbf{k}')}} \right) (\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}')) \\ &\qquad \times \sum_{j=1}^{N} \{ (e^{-i\alpha \mathbf{k} \cdot \mathbf{x}_j} - 1)(e^{-i\alpha \mathbf{k}' \cdot \mathbf{x}_j} - 1) \}, \end{aligned} \tag{I.17}$$

$$\begin{aligned} \mathbb{G}_{1,1}(\mathbf{k}, \lambda; \mathbf{k}', \lambda') &:= \left(\frac{\bar{\kappa}(\mathbf{k}) \, \bar{\kappa}(\mathbf{k}')}{\pi^2 \sqrt{\omega(\mathbf{k}) \, \omega(\mathbf{k}')}} \right) (\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}')^*) \\ &\qquad \times \sum_{j=1}^{N} \{ (e^{-i\alpha \mathbf{k} \cdot \mathbf{x}_j} - 1)(e^{i\alpha \mathbf{k}' \cdot \mathbf{x}_j} - 1) \}. \end{aligned} \tag{I.18}$$

The Hamiltonian $\mathbb{H}_2$ in (I.12) will serve as a starting point and a motivation in constructing the model treated in this paper. The crucial difference between our model and $\mathbb{H}_2$ is the presence of a spatial cutoff (i.e., a cut off in the $\sum \mathbf{x}_j$-variable) in the interaction term, $\mathbb{W}_g$. This spares us the trouble of controlling the large $\sum \mathbf{x}_j$-behaviour of the interactions. A spatial cutoff in the $\sum \mathbf{x}_j$-variable is superfluous, provided the potential $\mathbb{V}(x, \eta^{-1} R)$ grows faster than linearly in x at infinity, i.e., $\mathbb{V}$ confines the electrons to the nuclei. While the full model will be treated in a subsequent publication, we focus in this paper on the resulting simplified model. Because we only study the interactions between electrons *bound to static nuclei* and the quantized radiation field, the large-$\mathbf{x}_j$ cutoff is not expected to change the physics.

I.2. *Hamiltonians of Quantum Electrodynamics of Nonrelativistic Confined Particles*

In this section, we introduce the model to be treated in this paper. It is closely related to the standard model described in the last section, given by the Hamiltonian $\mathbb{H}_2$ in Eqns. (I.12)–(I.18). Compared to the latter model, the crucial difference is that its coupling functions have a better behaviour for large $|\mathbf{x}_j|$. Careful comparison of both models is done in Section I.6. Moreover, we introduce a few inessential simplifications which serve to streamline notations and to make the key ideas underlying our analysis more transparent.

First, we neglect electron spin and replace the photons by scalar particles. We thus redefine the Hilbert space of the system to be given by

$$\mathscr{H} = \mathscr{H}_{el} \otimes \mathscr{H}_f \tag{I.19}$$

where $\mathscr{H}_{el}$ is $L^2(X, dx)$ or a subspace of it,

$$\mathscr{H}_{el} \subseteq L^2(X, dx), \tag{I.20}$$

with $X := \mathbb{R}^{3N}$, for some $N \in \mathbb{N}$, being the particle configuration space. The Fock space of scalar photons is defined to be

$$\mathscr{H}_f \equiv \mathscr{F} := \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3, d^3k)^{\otimes_s n}. \tag{I.21}$$

We study Hamiltonians of the form

$$H_g = H_0 + W_g = H_{el} \otimes \mathbf{1} + \mathbf{1} \otimes H_f + W_g, \tag{I.22}$$

where H_{el} is a Schrödinger operator on $\mathscr{H}_{el}$,

$$H_{el} := -\Delta_x + V(x), \tag{I.23}$$

and the photon Hamiltonian, H_f, is now given by

$$H_f := \int dk\, \omega(k)\, a^\dagger(k)\, a(k) \qquad \text{with} \quad \omega(k) = |k|. \tag{I.24}$$

Here, we write k, instead of $\mathbf{k}$, for vectors $k \in \mathbb{R}^3$. We adopt this notation henceforth. The interaction, W_g, is of the form

$$W_g := g W_1 + g^2 W_2, \qquad \text{where} \tag{I.25}$$

$$W_1 := \int \{G_{1,0}(k) \otimes a^\dagger(k) + G_{0,1}(k) \otimes a(k)\}\, dk, \tag{I.26}$$

$$\begin{aligned} W_2 := \int \{ & G_{2,0}(k, k') \otimes a^\dagger(k)\, a^\dagger(k') + G_{0,2}(k, k') \otimes a(k)\, a(k') \\ & + G_{1,1}(k, k') \otimes a^\dagger(k)\, a(k')\}\, dk\, dk', \end{aligned} \tag{I.27}$$

where $G_{m,n}$ are functions of k or (k, k'), respectively, with values in the operators on $\mathscr{H}_{el}$.

Next, before adding further specifications about W_g, we discuss spectral properties of H_0 and the problem of perturbation, H_g when $g \neq 0$. We formulate the main questions about mathematical models of radiation, and we briefly describe our answers to those questions.

We assume that the particle Hamiltonian H_{el} in (I.23), in the absence of the radiation field, is selfadjoint and has a standard spectrum, i.e., a continuum corresponding to the half-axis $[\Sigma, \infty)$, for some $\Sigma \leqslant 0$, and discrete eigenvalues $E_0, E_1, ...$ on the left of (below) the continuum, i.e., $E_0 < E_1 < \cdots < \Sigma$ (HVZ Theorem, see e.g. [37]). The eigenvalues, E_j, can, of course, be degenerate (see Fig. I.1).

The spectrum of H_f consists of a simple eigenvalue at 0 (corresponding to the vacuum vector, $\Omega \in \mathscr{F}$) and absolutely continuous spectrum covering the half-axis $[0, \infty)$. Consequently, by separation of variables, the unperturbed Hamiltonian

$$H_0 = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f, \tag{I.28}$$

has the same point spectrum as H_{el}, i.e., $\{E_j\}$, and the continuum covers the half-axis $[E_0, \infty)$ and consists of a union of branches, $[E_j, \infty)$, starting at the eigenvalues E_j, and the branch $[\Sigma, \infty)$ (see Fig. I.2).

The crucial fact is that the eigenvalues E_j lie at tips of branches of continuous spectrum of H_0, i.e., the numbers E_j play a double role as eigenvalues and as thresholds of continuous spectrum.

The thresholds of continuous spectrum of H_0 are the branch points of the Riemann surface of $f_{u,v}(z) = \langle u | (H_0 - z)^{-1} v \rangle$, for arbitrary u, v in a

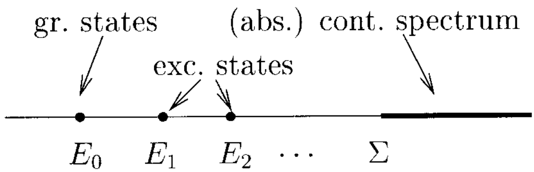

FIG. I.1. The spectrum of H_{el}.

certain dense set of $\mathscr{H}$. Figure I.2 describes a particular projection of this Riemann surface onto the complex energy plane. A different projection is shown in Fig. I.3 (see Section I.5 for more details).

The eigenfunctions of H_0 corresponding to the eigenvalues E_j have the form $\psi_j \otimes \Omega$, where ψ_j are the eigenfunctions of H_{el} corresponding to the eigenvalues E_j, and Ω is the vacuum in $\mathscr{F}$, $H_f \Omega = 0$. The fate of these eigenfunctions and the corresponding eigenvalues is a key concern in the theory of radiation.

Mathematically, we are faced with a problem of perturbation of eigenvalues located at thresholds of continuous spectrum. This is the most difficult situation in perturbation theory. (To our knowledge, it has not been treated in the literature, yet.) The fact that the eigenvalues $\{E_j\}$ of the atomic Hamiltonian H_{el} are also branch points of continuous spectrum of H_0 is due to the property of photons that their *rest mass vanishes*, i.e., $\omega(0) = 0$. (From this it follows that the continuous spectrum of H_f covers the entire half-axis $[0, \infty)$, and thus $[E_j, \infty)$ is a branch of continuous spectrum of H_0.) The ensuing difficulties in developing a convergent perturbation theory for the eigenvalues E_j are an aspect of the celebrated *infrared problem* of quantum electrodynamics. The fact that the number of photons is neither conserved nor bounded, for any energy interval, leads to an infinite number of degrees of freedom in the perturbation problem. In physics language, small fluctuations of energy allowed by the uncertainty principle ($\Delta E\, \Delta t \geqslant \hbar$) may produce an infinite number of photons whose energy is very close to 0. Such photons are called *soft photons.* A standard physical imagery is of an atom surrounded by a cloud of soft photons. The interaction between charged particles and the quantized radiation field leads to a renormalization of physical parameters. In particular, it produces the energy shifts first measured by Lamb and Retherford and estimated by Bethe [7, 8] (the Lamb shift and radiative corrections).

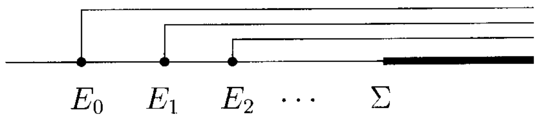

FIG. I.2. The Spectrum of $H_0 = H_{el} \otimes \mathbf{1} + \mathbf{1} \otimes H_f$.

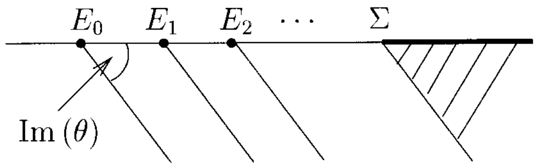

FIG. I.3. A projection of the Riemann surface of $\langle u|(H_0-z)^{-1}v\rangle$ onto the energy plane.

The main *physical* expectations underlying the theory of radiation are that there exists a stable ground state and that the excited states of the particle system, corresponding to the eigenvalues E_j, with $j>0$, become unstable. They decay spontaneously (i.e., acquire a finite life-time) by emission of photons. Mathematically, this phenomenon is described in the language of *resonances* or, more generally, in terms of the Riemann surface of matrix elements of the resolvent, $\langle u \mid (H_g-z)^{-1} v\rangle$, for a universal dense set of vectors containing u, v.

Next we formulate some key mathematical problems in the theory of radiation.

(a) Is the ground state (corresponding to the eigenvalue E_0) of H_0 stable when the interactions of particles with the quantized radiation field are included?

(b) Are the excited states corresponding to the eigenvalues E_j, $j>0$, of H_0 unstable?

(c) Do the excited states turn into resonances? If they do, what are the positions (i.e., the energies and life-times) of these resonances in the complex energy plane?

(d) What is the structure of the Riemann surface of $\langle u \mid (H_g-z)^{-1} v\rangle$ (for a universal dense set $\mathcal{D}\ni u$, v, see [pp. 54–55, 37])?

The main results of this paper are as follows.

(i) H_g has a ground state originating from the ground state of H_0. (The coupling constant g does not need to be particularly small, for this result to hold.) The ground state is exponentially localized in the particle coordinates.

(ii) H_g has no other eigenvalues outside small neighbourhoods of the thresholds of H_{el}, besides the ground state energy. In other words, the excited states of H_g become unstable. This is proved for sufficiently small coupling constants. Moreover, the spectrum of H_g outside $O(g^2)$-neighbourhoods of the thresholds of H_{el} is *purely absolutely continuous.*

(iii) There are complex "eigenvalues" $\{E_{j,\ell}(g)\}_{\ell=1}^{N_j}$ bifurcating from each eigenvalue E_j of H_0, where N_j is the multiplicity of E_j. The energy

$E_{0,1}(g)$ with the *smallest* real part bifurcating from the ground state energy E_0 of H_g is *real* and is the ground state energy of H_g (see result (i)). Under certain genericity assumptions on the coupling functions, $G_{m,n}$ (see (I.49)), all energies $E_{j,\ell}(g)$, $j \geqslant 1$, have a negative imaginary part, for $g \neq 0$. They are the complex resonance energies of H_g. (These notions will be explained more precisely, below.)

(iv) We develop a constructive algorithm to calculate the functions $E_{j,\ell}(g)$. In particular, the radiative corrections, $E_{j,\ell}(g) - E_j(0)$, are given by

$$E_{j,\ell}(g) - E_j(0) = \varepsilon_{j,\ell}\, g^2 + o(|g|^2), \tag{I.29}$$

where $\mathrm{Re}[\varepsilon_{j,\ell}]$ is given by *Bethe's formula* and $\mathrm{Im}[\varepsilon_{j,\ell}]$ is given by *Fermi's Golden Rule* (as will be explained and proven below and in [5]). For the hydrogen atom, the energy differences $\mathrm{Re}[(E_{j,\ell}(g) - E_{i,k}(g)) - (E_j(0) - E_i(0))]$ reproduce the observed Lamb shifts quite accurately, (i.e., up to effects due to *the ultraviolet cutoff and relativistic corrections*).

(v) Let $C_0(\mathbb{R}^3)$ denote the space of continuous functions on $\mathbb{R}^3$ of compact support, and let $\mathscr{F}(C_0(\mathbb{R}^3))$ be the photon Fock space over $C_0(\mathbb{R}^3)$, i.e.,

$$\mathscr{F}(C_0(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} C_0(\mathbb{R}^3)^{\otimes_s n}. \tag{I.30}$$

For all $u, v \in C_0(X) \otimes \mathscr{F}[C_0(\mathbb{R}^3)]$, $\langle u | (H_g - z)^{-1} v \rangle$ has an analytic continuation in the variable z into the part of the lower complex half-plane depicted in Fig. I.4.

The complex numbers $E_{j,\ell}(g)$ are singularities of this continuation. The angle of inclination of the cuts emanating from $E_{j,\ell}(g)$ is related to the choice of projection of the Riemann surface, on which $\langle u, (H_g - z)^{-1} v \rangle$ is defined, onto the energy plane. The numbers $E_{j,\ell}(g)$ are *independent* of the choice of projection. For $j \geqslant 1$, they correspond to resonances of H_g.

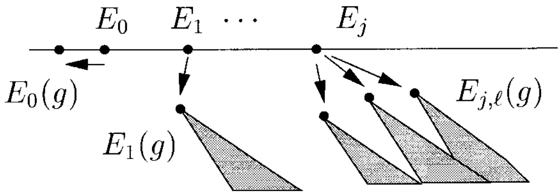

FIG. I.4. A projection of the Riemann surface for $\langle u | (H_g - z)^{-1} v \rangle$.

the particle system was replaced by a harmonic oscillator, and in [15, 18–20, 24–26, 41], the particle system was replaced by a 2×2 matrix (the spin-boson model) or by a finite-rank matrix. The existence of a ground state for the spin-boson model without infrared cutoff was proved by a method and under conditions different to ours in [41], and recently, a proof of that fact by methods and conditions similar to ours was given in [3]. The results of the present paper were announced in [4].

In our analysis, we use the following tools: Our proof of (i) involves constructive-field-theory techniques developed in [13, 16]. Our proof of (ii) involves positive-commutator techniques (see also [15, 20] and [4]) and the technique of complex dilatation, in conjunction with the isospectral Feshbach map which is a far-reaching extension of the standard Feshbach projection method. A powerful extension of the positive-commutator method appears in [6].

The deepest results of this paper are (iii) and (iv). In order to establish them, we develop an operator-theoretic *renormalization group method* for non-selfadjoint Hamiltonians, based on an iterative use of the isospectral Feshbach map. Our renormalization map acts directly on quantum Hamiltonians, rather than on their correlation functions, or indirectly through partition functions, as is the case with the standard approach.

As a biproduct, our renormalization group construction yields alternative proofs of results (i) and (ii), for small values of $|g|$.

I.3. *Main Results*

In this section, we present a precise formulation of our assumptions and of our main results. Recall from (I.22)–(I.24) that $H_g = H_0 + W_g$, with $H_0 = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f$. The operator $H_f = \int d^3k\, \omega(k)\, a^\dagger(k)\, a(k)$, where $\omega(k) = |k|$, is a selfadjoint operator on its natural dense domain, $\mathscr{D}(H_f) \subseteq \mathscr{F}$. It is positive and

$$\sigma(H_f) = \sigma_{\mathrm{pp}}(H_f) \cup \sigma_{\mathrm{ac}}(H_f),$$

where $\sigma_{\mathrm{pp}}(H_f)$ consists of the simple eigenvalue 0, corresponding to the eigenvector $\Omega \in \mathscr{F}$ (the vacuum); $\sigma_{\mathrm{ac}}(H_f)$ is the absolutely continuous spectrum of H_f. It covers the half-axis $[0, \infty)$ and has infinite multiplicity. The singular continuous spectrum of H_f is empty.

Next, $H_{el} = -\Delta_x + V(x)$ is a Schrödinger operator. Throughout the paper we assume *that H_{el} is selfadjoint and has at least one eigenvalue, E_0, below its continuum, Σ*. More specifically, we assume that H_{el} is an N-body Hamiltonian, i.e., that the potential $V(x)$ obeys:

$V(x)$ is a real function of the form $\sum_{\mathrm{finite}} V_i \circ \pi_i$, where π_i are linear maps from X $(= \mathbb{R}^{3N}$, the particle configuration space$)$ to $\mathbb{R}^{n_i}$ and $V_i \in L^{p_i}(\mathbb{R}^{n_i}) + L^\infty(\mathbb{R}^{n_i})$, with $p_i = 2$ if $n_i \leqslant 3$, $p_i > 2$ if *$n_i = 4$, and $p_i = n_i/2$ if $n_i \geqslant 5$.*

This is not the most general condition that suits our purposes, but it is simple and general enough, as it includes $\mathbb{V}$ in (I.7). In particular, under this condition H_{el} is selfadjoint on the domain $\mathscr{D}(H_{el}) = \mathscr{D}(-\Delta_x)$. From our Hypothesis on $V(x)$ above and the HVZ-Theorem (see e.g. [10, 37]), it follows that the particle Hamiltonian H_{el} in (I.23) has a standard spectrum: a continuum corresponding to the half-axis $[\Sigma, \infty)$, for some $\Sigma \leqslant 0$, and discrete eigenvalues E_j of finite multiplicity, $1 \leqslant N_j < \infty$, $j = 0$, 1, 2, ... on the left of (below) the continuum, i.e., $E_0 < E_1 < \cdots < \Sigma$.

Our model is specified by conditions on the coupling functions $G_{m,n}$ in the interaction W_g defined in (I.25)–(I.27). For different results of this paper we need different conditions. We list these conditions below. For notational convenience the hypotheses presented below tend to be somewhat stronger than those used in the actual proofs. More general hypotheses can be found in the corresponding chapters; (see the end of this chapter for a guide). Here is the first set of conditions on $G_{m,n}$ that we use in Chapters II and III:

HYPOTHESIS 1. *The coupling functions $G_{2,0} \equiv G_{0,2} \equiv G_{1,1} \equiv 0$ vanish identically, and $G_{0,1}(k) = G_{1,0}(k)^*$ is a multiplication operator in the Schrödinger representation $\mathscr{H}_{el} = L^2(X)$, i.e., for every $k \in \mathbb{R}^3$, $G_{0,1}(k)$ acts as multiplication by a function which is denoted $G_x(k)$. This coupling function $G_x(k)$ obeys the following bounds, for some $\varepsilon > 0$:*

$$\Lambda_1 := \sup_{x \in \mathbb{R}^{3N}} \left\{ \int \left[1 + \frac{1}{\omega(k)} \right] |G_x(k)|^2 \, dk \right\}^{1/2} < \infty, \tag{I.33}$$

$$\Lambda_{2,\varepsilon} := \sup_{x \in \mathbb{R}^{3N}} \left\{ \mathrm{e}^{-\varepsilon |x|} \int \frac{|G_x(k)|^2}{\omega(k)^2} \, dk \right\}^{1/2} < \infty. \tag{I.34}$$

We refer to W_1 as W_x.

The second set of hypotheses is somewhat more general than Hypothesis 1. It is used in Chapters IV and V (confer to the end of Section I.1).

HYPOTHESIS 2. *For some $\theta_0 > 0$ and all $k, k' \in \mathbb{R}^3$, the maps*

$$\theta \mapsto (-\Delta_x + 1)^{-1/4} \, \mathrm{e}^{3\theta/2} G_{m,n}(\mathrm{e}^{-\theta} k)(-\Delta_x + 1)^{-1/4}, \tag{I.35}$$

for $m + n = 1$, and

$$\theta \mapsto \mathrm{e}^{3\theta} G_{m,n}(\mathrm{e}^{-\theta} k, \mathrm{e}^{-\theta} k'), \tag{I.36}$$

for $m+n=2$, *are analytic as maps:* $\{|\theta| \leqslant \theta_0\} \to \mathscr{B}[\mathscr{H}_{el}]$, *the bounded operators on* $\mathscr{H}_{el}$. *Define* $J(k)$ *to be the smallest non-negative function such that, for* $m+n=1$,

$$\sup_{|\theta| \leqslant \theta_0} \|(-\Delta_x+1)^{-1/4} \, e^{3\theta/2} G_{m,n}(e^{-\theta}k)(-\Delta_x+1)^{-1/4}\| \leqslant J(k), \tag{I.37}$$

and, for $m+n=2$,

$$\sup_{|\theta| \leqslant \theta_0} \|e^{3\theta/2} G_{m,n}(e^{-\theta}k, e^{-\theta}k')\| \leqslant J(k)\, J(k'). \tag{I.38}$$

Then J *obeys*

$$\Lambda_{1+\beta} := \left\{ \int [1+\omega(k)^{-1-\beta}] \, |J(k)|^2 \, dk \right\}^{1/2} < \infty, \tag{I.39}$$

for some $\beta > 0$.

In Section V, where our renormalization group analysis is presented, we need the following assumption, in addition to Hypothesis 2:

HYPOTHESIS 3. *Hypothesis* 2 *holds. The function* J, *defined by* (I.37) *and* (I.38), *obeys*

$$\Lambda(\mu) := \sup_{k \in \mathbb{R}^3} \{\omega(k)^{(1-\mu)/2} \, J(k)\} < \infty, \tag{I.40}$$

for some $\mu > 0$.

The condition that μ be strictly positive ensures that we are dealing with a perturbation problem that is *asymptotically free* in the *infrared region* of the photon degrees of freedom. To see this, we assume, for simplicity, that the electron degrees of freedom have already been eliminated (by taking a "partial trace" over $\mathscr{H}_{el}$). The resulting effective Hamiltonian, H_{eff}, then has the form $H_{\text{eff}} = E + T[H_f] + \sum_{m+n \geqslant 1} W_{m,n}$ with $E \in \mathbb{C}$, $T[r] - r = O(g^2 r)$, and

$$W_{m,n} = \int a^\dagger(k^{(m)}) \, w_{m,n}[H_f; k^{(m)}; \tilde{k}^{(n)}] \, a(\tilde{k}^{(n)}) \, dk^{(m)} \, d\tilde{k}^{(n)},$$

where

$$k^{(m)} := (k_1, ..., k_m), \qquad dk^{(m)} := \prod_{j=1}^{m} d^3 k_j, \qquad \text{and} \qquad a^\dagger(k^{(m)}) := \prod_{j=1}^{m} a^\dagger(k_j).$$

Hypotheses 2 and 3 turn out to imply that

$$\sup_{r \geqslant 0} |w_{m,n}[r; k^{(m)}; \tilde{k}^{(n)}]| \leqslant \xi^{m+n} \prod_{j=1}^{m} |k_j|^{(\mu-1)/2} \prod_{j=1}^{n} |\tilde{k}_j|^{(\mu-1)/2}, \tag{I.41}$$

for some constant ξ and all $m, n \in \mathbb{N}_0$ such that $m+n \geqslant 1$.

A key problem arising in our renormalization group analysis is to estimate the size of matrix elements of H_{eff} between vectors in the subspace $\chi_{H_f < \rho} \mathscr{F}$ of photon Fock space, where $\chi_{H_f < \rho}$ is the spectral projection of H_f onto energies smaller than ρ, for an *arbitrary photon energy scale* ρ, with $0 < \rho < \rho_0$. Here, the constant ρ_0 is of order $O(1$ *Rydberg*$)$. We choose units such that $\rho_0 = 1$. By dilating the photon momenta

$$k \to \rho k,$$

we find that H_{eff} is unitarily equivalent to the Hamiltonian $\rho H_{\text{eff}}^{(\rho)}$, where

$$H_{\text{eff}}^{(\rho)} = \rho^{-1} E + T^{(\rho)}[H_f] + \sum_{m+n \geqslant 1} \rho^{(1+(\mu/2))(m+n)-1} W_{m,n}^{(\rho)}, \tag{I.42}$$

with $T^{(\rho)} := \rho^{-1} T[\rho H_f]$ and

$$W_{m,n}^{(\rho)} = \int a^\dagger(k^{(m)})\, w_{m,n}^{(\rho)}[H_f; k^{(m)}; \tilde{k}^{(n)}]\, a(\tilde{k}^{(n)})\, dk^{(m)}\, d\tilde{k}^{(n)}, \tag{I.43}$$

$$w_{m,n}^{(\rho)}[r; k^{(m)}; \tilde{k}^{(n)}] := \rho^{(\mu-1)(m+n)/2}\, w_{m,n}[\rho r; \rho k^{(m)}; \rho \tilde{k}^{(n)}]. \tag{I.44}$$

Using the bounds (I.41), we find that the kernels $w_{m,n}^{(\rho)}$ again satisfy the bounds (I.41). Because $m+n \geqslant 1$, and since $\mu > 0$ (by Hypothesis 3), each term in the perturbation of H_f on the right side of (I.43) is multiplied by a factor bounded above by

$$\rho^{\mu(m+n)/2} \tag{I.45}$$

which tends to 0 exponentially fast in $m+n$, as $\rho \to 0$. This bound implies that the perturbations $W_{m,n}$ are irrelevant [11, 29] in the infrared.

More specifically, this feature will imply that the effective Hamiltonian, on a photon energy scale $0<\rho<1$, of the physical systems we study, is a small perturbation of the operator

$$T^{(\rho)}[H_f]+\Delta E^{(\rho)}\mathbf{1}, \tag{I.46}$$

with norm bounded by $O(\rho^{\mu/2})$. Moreover,

$$T^{(\rho)}[r]\to r, \qquad \rho\,\Delta E^{(\rho)}\to\Delta E^{(0)},$$

as $\rho\to 0$; (see Section V).

When $\mu=0$, $W_{0,1}$ and $W_{1,0}$ are marginal perturbations, while $W_{m,n}$ are irrelevant, for $m+n\geqslant 2$. Then the renormalization group analysis becomes more subtle. We defer these problems to a separate analysis.

Our results concerning the Hamiltonian H_g, defined in (I.22)–(I.27) are summarized in the following three theorems.

THEOREM I.1 (Binding). *Suppose that Hypothesis* 1 *holds for some* $0\leqslant \varepsilon<(\Sigma-E_0-g^2\ |G_x|^2/\omega)^{1/2}$. *Then* $E_0(g):=\inf G(H_g)>-\infty$, *and*

(a) *There exists a constant,* $M_\varepsilon<\infty$, *which depends only on* V *and* Λ_1, *such that if*

$$\delta_\varepsilon:=g^2M_\varepsilon\Lambda^2_{2,\varepsilon}\cdot\left[1+4g^2(\Sigma-E_0)^{-2}\sup_x\left\{\int|G_x(k)|^2\,dk\right\}\right]<1, \tag{I.47}$$

then the Hamiltonian H_g *has a ground state* $\phi\in\mathscr{H}$, $H_g\phi=E_0(g)\phi$.

(b) *This ground state has a non-vanishing overlap with* $P_{el}\otimes P_\Omega$, *where* $P_{el}:=\chi_{H_{el}\leqslant(E_0+\Sigma)/2}$ *is the (finite dimensional) projection onto the bound states of* H_{el} *with energy below* $(E_0+\Sigma)/2$, *and* $P_\Omega=|\Omega\rangle\langle\Omega|$ *is the projection onto the photon vacuum. More specifically,* $\langle P_{el}\otimes P_\Omega\rangle_\phi\geqslant 1-\delta_\varepsilon>0$.

(c) *Moreover, this ground state satisfies the exponential bound*

$$\|e^{\varepsilon|x|}\otimes\mathbf{1}_f\,\phi\|\leqslant M_\varepsilon. \tag{I.48}$$

(d) *As* $g\to 0$, *the ground state energy and the ground state subspace of* H_g *converge to the unperturbed ground state energy (= the ground state energy of the atom or molecule) and the ground state subspace, respectively.*

(e) *If* $\mathscr{H}_{el}=L^2(\mathbb{R}^{3N})$ *and* $G_x(k)=\overline{G_x(-k)}$, *for all* $k\in\mathbb{R}^3$, *then the ground state is unique.*

(f) *Assume Hypotheses* 2 *and* 3 *with* $\theta_0=0$ *hold. Set* $\tau:=\min\{E_1,\Sigma\}-E_0$. *Then, for* $g\tau^{-1/2}$ *sufficiently small, the degeneracy of the ground state of* H_g *does not exceed the degeneracy of the ground state of* H_{el}.

THEOREM I.2 (Instability of Excited States). *Suppose Hypothesis 2 holds, and assume that $E_j > E_0$ is an isolated eigenvalue of H_{el} of degeneracy $N_j < \infty$ with eigenvectors $\psi_{j,1}, ..., \psi_{j,N_j}$. Assume that the $N_j \times N_j$ selfadjoint matrix*

$$(A_j)_{\mu\nu} := \int_{-\infty}^{E_j - 0} \int dk \, \langle G_{10}(k) \, \psi_{j,\mu} \, | \, d\chi_{H_{el} \leqslant E} G_{10}(k) \, \psi_{j,\nu} \rangle \; \delta[\omega(k) - E_j + E] \tag{I.49}$$

is positive definite, i.e., $A_j \geqslant a_j \mathbf{1} > 0$. Then we have that

(a) *the operator H_g has absolutely continuous spectrum in each interval*

$$[\tfrac{2}{3} E_{j-1} + \tfrac{1}{3} E_j, \tfrac{1}{3} E_j + \tfrac{2}{3} E_{j+1}];$$

(b) *for vectors $u, v \in \mathscr{D}$, the matrix elements $\langle u | (H_g - z)^{-1} v \rangle$ of the resolvent $(H_g - z)^{-1}$ have an analytic continuation from $\mathbb{C}_+$ into*

$$\mathscr{I}_j := [\tfrac{2}{3} E_{j-1} + \tfrac{1}{3} E_j, \tfrac{1}{3} E_j + \tfrac{2}{3} E_{j+1}] + i(\mathbb{R}_+ - \gamma_j), \tag{I.50}$$

where $\gamma_j := g^2 a_j$.

In particular, excited states of H_{el} are unstable under the perturbation W_g.

Let $\mathbb{C}_\pm$ denote the upper and lower halfplane, respectively.

THEOREM I.3 (Riemann Surface of H_g). *Suppose that Hypotheses 2 and 3 hold, let $E_0(g) = \inf \sigma(H_g)$, and pick $\tau > 0$. Then, for some $\eta, \vartheta > 0$, for g sufficiently small, and for $u, v \in C_0(X) \otimes \mathscr{F}(C_0(\mathbb{R}^3))$, $\langle u | (H_g - z)^{-1} v \rangle$ has an analytic continuation across the interval $\mathscr{I} := (E_0(g), \Sigma - \tau) \subset \sigma_{\text{cont}}(H_g)$ into the domain $\{\mathscr{I} + i(-\eta + \mathbb{R}_+)\} \setminus \mathscr{A}$, where*

$$\mathscr{A} := \bigcup_{j \geqslant 0} \bigcup_{\ell = 1}^{N_j} \{E_{j,\ell}(g) + T_{j,\ell}[g, e^{-i\vartheta} r] + b \, | \, r > 0, |b| \leqslant r^{1 + (\mu/2)}\}, \tag{I.51}$$

contained in the second Riemann sheet (see Fig. I.5). Here $E_{j,\ell}(g)$ and $T_{j,\ell}[g, \cdot]$ have the following properties, for all $j \geqslant 0$ and $1 \leqslant \ell \leqslant N_j$.

(a) *$E_{j,\ell}(g) \in \overline{\mathbb{C}_-}$ and $E_{j,\ell}(g) \to E_j$, as $g \to 0$,*

(b) *$T_{j,\ell}[g, \cdot] \in C^0(\mathbb{R}_+)$ and $|\partial_r T_{j,\ell}[g, r] - 1| \leqslant 1/8$,*

(c) *$E_{0,1}(g)$ is the ground state energy, and $E_{j,\ell}(g)$ are resonances of H_g (see Section I.5 for precise definitions), provided $j \geqslant 1$ and A_j in (I.49) is strictly positive definite.*

(d) *$E_{j,\ell}(g)$ and $T_{j,\ell}[g,\cdot]$ can be explicitly computed to any order in g by a convergent renormalization group scheme. In particular, we have*

$$E_{j,\ell}=E_j+e_{j,\ell}\,g^2+o(g^2), \tag{I.52}$$

where $\mathrm{Re}\{e_{j,\ell}\}\,g^2$ *is given by Bethe's formulae for radiative corrections, yielding the Lamb shift. Furthermore,* $\mathrm{Im}\{e_{j,\ell}\}$ *is an eigenvalue of the matrix A_j, generalizing Fermi's golden rule to the degenerate case* (*see Figure* I.5).

Before proceeding to some simple technical matters, we give an outline of the *organization* of this paper. Our paper contains five chapters. The present chapter is an introductory one. The remainder of this chapter is devoted to the derivation of some elementary bounds, proving e.g. the self-adjointness of H_g, and to the definition and discussion of the complex dilatations of H_g needed in the proof of Theorems I.2 and I.3. We will conclude this chapter with a comparison of Hypotheses 1–3 to features of the standard model of nonrelativistic QED, as described by the Hamiltonian $\mathbb{H}_2$ in (I.10)–(I.18). Chapters II–V are devoted to an analysis of the operator H_g. Chapter II, Chapter III and Sections IV–V are fairly independent of each other.

Chapter II deals with the problem of binding. Its main result is Theorem I.1(a) which is a transcription of Theorem II.8 and Corollary II.9.

Chapter III deals with the nature of the spectrum of H_g. Here, we establish positivity of certain commutators. The main result in this chapter is Corollary III.5. Improved results of this type appear in [6].

In Chapter IV, Section IV.1, the reader will have the first glimpse at what we term the (isospectral) Feshbach map. The Feshbach map is the main connection of Chapter IV with Chapter V. Using the Feshbach map, in conjunction with complex dilatation, we derive Fermi's golden rule in Section IV.2. Here, our main result is Theorem IV.3 which immediately implies Theorem I.2.

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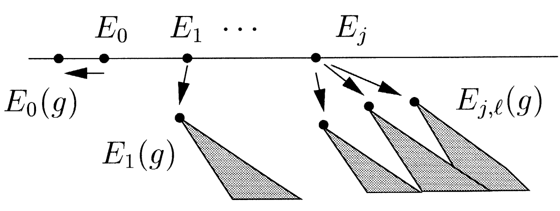

FIG. I.5. A projection of the Riemann surface for $\langle u|(H_g-z)^{-1}v\rangle$.

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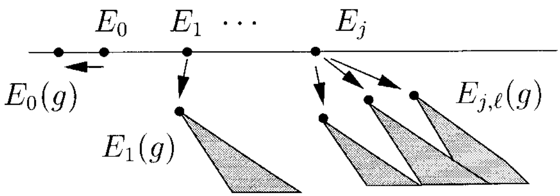


FIG. I.5. A projection of the Riemann surface for $\langle u|(H_g - z)^{-1} v\rangle$.

Proof. Let $f_x = G_x/\omega$. Then, omitting the integration variable k,

$$\mathbf{1}_{el} \otimes H_f + gW_x = \int [\mathbf{1}_{el} \otimes a^\dagger a + f_x \otimes a^\dagger + f_x^* \otimes a]\,\omega\,dk$$

$$= \int [(\mathbf{1}_{el} \otimes a + f_x \otimes \mathbf{1}_f)^* (\mathbf{1}_{el} \otimes a + f_x \otimes \mathbf{1}_f)]\,\omega\,dk$$

$$- \int \frac{|G_x|^2}{\omega} \otimes \mathbf{1}_f.$$

Hence

$$\mathbf{1}_{el} \otimes H_f + gW_x \geqslant -\int \frac{|G_x|^2}{\omega} \otimes \mathbf{1}, \tag{I.55}$$

which implies (I.54). ∎

LEMMA I.6 (Relative Bounds).

$$\|a(f)\psi\|_{\mathscr{F}} \leqslant \left(\int \frac{|f|^2}{\omega} \right)^{1/2} \|H_f^{1/2}\psi\|_{\mathscr{F}} \tag{I.56}$$

and

$$\|a^\dagger(f)\psi\|_{\mathscr{F}}^2 \leqslant \int \frac{|f|^2}{\omega} \|H_f^{1/2}\psi\|_{\mathscr{F}}^2 + \int |f|^2 \|\psi\|_{\mathscr{F}}^2 \tag{I.57}$$

Proof. By the Schwarz inequality we have (dropping the subscript $\mathscr{F}$)

$$\|a(f)\psi\| \leqslant \int |f|\,\|a\psi\| \leqslant \left(\int \frac{|f|^2}{\omega} \right)^{1/2} \left(\int \omega\,\|a\psi\|^2 \right)^{1/2}.$$

Since

$$\int \omega\,\|a\psi\|^2 = \langle \psi, H_f\psi \rangle,$$

this implies (I.56). Next, we use that

$$a(f)\,a^\dagger(f) = \iint \bar{f}(p)\,f(k)\,a(p)\,a^\dagger(k) = \iint \bar{f}(p)\,f(k)\,a^\dagger(k)\,a(p) + \int |f(p)|^2,$$

and that

$$\iint \bar{f}(p)\, f(k) \langle a^\dagger(k)\, a(p) \rangle_\psi \leqslant \left(\int f(p)\, \|a(p)\psi\|^2 \right)^2 \leqslant \int \frac{|f|^2}{\omega} \langle H_f \rangle_\psi,$$

to obtain (I.57). ∎

Since the interaction W_x can be written as

$$W_x = a^\dagger(G_x) + a(G_x), \tag{I.58}$$

we have the following corollary.

COROLLARY I.7 (Relative Bound of W_x w.r. to $(H_f)^{1/2}$). *Suppose that Hypothesis* 1 *holds; in particular, assume* $\Lambda_1 = \sup_x \int |G_x|^2 + \sup_x \int |G_x|^2/\omega < \infty$. *Then*

$$\|W_x \psi\|^2 \leqslant 2 \left\| \left(\int \frac{|G_x|^2}{\omega} \right) \otimes H_f^{1/2} \psi \right\|^2 + \left\| \int |G_x|^2 \otimes \mathbf{1}_f \psi \right\|^2. \tag{I.59}$$

Moreover,

$$|\langle W_x \rangle_\psi| \leqslant 2 \sup_x \left\{ \int \frac{|G_x|^2}{\omega} \right\}^{1/2} \cdot \|H_f^{1/2} \psi\| \cdot \|\psi\|. \tag{I.60}$$

Thus H_g is essentially selfadjoint on the domain $D(H_{el}) \otimes D(H_f)$.

It is interesting to extend this result to Hamiltonians obeying Hypothesis 2.

COROLLARY I.8 (Relative bound of W_l). *Assume Hypothesis* 2 *with* $\theta = 0$. (*In particular, no analyticity is required and* $\theta = 0$ *in* (I.35)–(I.36)). *Then*

$$\|W_1 \psi\| \leqslant 2\Lambda_{1+\beta}\, \|(-\Delta + 1)^{1/2} \otimes (H_f + 1)^{1/2} \psi\|, \tag{I.61}$$

$$\|W_2 \psi\| \leqslant 2\Lambda_{1+\beta}\, \|\mathbf{1}_{el} \otimes (H_f + 1) \psi\|. \tag{I.62}$$

Thus, for $g < (4\Lambda_{1+\beta})^{-1}$, H_g is essentially selfadjoint on the domain $D(H_{el}) \otimes D(H_f)$.

I.5. *Resonances and Spectral Deformation*

The natural extension of the notion of eigenvalue is the notion of (quantum) resonance. Though originally not rigorously defined, it has played a key role in physics since the birth of quantum mechanics. The first rigorous definition of a resonance was given in [40]. Resonances were the subject of intensive study ever since this work appeared; (see the review in [39]). There are three ways to generalize the notion of an eigenvalue: starting from an eigenequation, from the space-time picture, or from the analytic structure of the resolvent. We use the third way to define a resonance, the first one as an efficient tool in investigating it, and the second one to interpret it physically.

We recall a definition from Section I.3. Let $C_0(\mathbb{R}^3)$ be the one-photon space of continuous and compactly supported functions. (In the vector model, functions on $\mathbb{R}^3$ should be replaced by transverse vector fields on $\mathbb{R}^3$, i.e., by vector fields, f, obeying $k \cdot f(k) = 0$.) Let $\mathscr{F}(C_0)$ be the bosonic Fock space over $C_0(\mathbb{R}^3)$, i.e., $\mathscr{F}(C_0) = \bigoplus_{n=0}^{\infty} C_0(\mathbb{R}^3)^{\otimes_s n}$, where $\otimes_s$ stands for the symmetric tensor product. This space is a core for H_f, while $\mathscr{D} \equiv C_0^\infty(X) \otimes \mathscr{F}(C_0)$ is a core for H_g. Consider the analytic continuation of $\langle u | (H_g - z)^{-1} v \rangle$, for $u, v \in \mathscr{D}$, from $\mathbb{C}_+$ (the upper half-plane) across the real axis into the lower half-plane, provided such a continuation exists. We define the *resonance eigenvalues* (or simply, *resonances*) as positions of singularities on the second sheet, i.e., in the lower half-plane $\mathbb{C}_-$, of the analytic continuation described above.

The connection with eigenvalues is simple. The eigenvalues of H_g, being positions of the singularities of $\langle u | (H_g - z)^{-1} v \rangle$ on the real axis, can turn into resonances under perturbations. The real part of the resonance is called the resonance energy, the imaginary part is the resonance width. The resonance width is interpreted as the decay rate, its reciprocal is called the life-time of the resonance.

In Schrödinger quantum mechanics, these singularities are isolated poles. But, for the Hamiltonians studied in this paper, they are located at branch points.

Next we proceed to explain the connection between our definition of resonances and an eigenvalue problem.

THEOREM I.9. *Assume Hypothesis* 2. *Then*

(i) $\langle u | (z - H_g)^{-1} v \rangle$ *has an analytic continuation from* $\mathbb{C}_+$ *across the continuous spectrum of* H_g *into a complex neighbourhood of* $\mathbb{R}$, *for all* $u, v \in \mathscr{D}$,

(ii) *there is a type-A family,* $\theta \mapsto H_g(\theta)$, *of operators analytic in a neighbourhood of* $\theta = 0$, *such that* $H_g(\theta)^* = H_g(\bar{\theta})$, $\sigma(H_g(\theta)) \subset \overline{\mathbb{C}_-}$, *for* $\operatorname{Im} \theta > 0$, *and*

$$H_g(\theta) = U(\operatorname{Re} \theta)\, H_g(\operatorname{Im} \theta)\, U(\operatorname{Re} \theta)^{-1}$$

for a one-parameter group, $U(\lambda)$, *of unitary dilatations. Additionally assuming Hypothesis* 3, *the singularities and branch points of the analytic continuation, described in* (i), *occur at the eigenvalues of* $H_g(\theta)$ *and are fixed points of* $\sigma[H_g(\theta)]$ *under small variations of* $\operatorname{Im} \theta > 0$.

Theorem I.9 follows from the results in Chapters IV and V. Here, we construct the family $H_g(\theta)$ described in the theorem. Let U_θ be a dilatation transformation on the one-photon space, i.e., $U_\theta\colon f(k) \to \mathrm{e}^{-3\theta/2} f(e^{-\theta} k)$. (More generally, one can shift along a flow in $\mathbb{R}^3$, see below.) Define the spectral deformation on the Fock space by

$$U_f(\theta)\, a^\dagger(f)\, U_f(\theta)^{-1} = a^\dagger(U_\theta f). \tag{I.63}$$

An application of Eq. (I.63) yields

$$H_f(\theta) = U_f(\theta)\, H_f\, U_f(\theta)^{-1} = \int dk\, \omega_\theta(k)\, a^\dagger(k)\, a(k),$$

where $\omega_\theta = U_\theta \omega(k)\, U_\theta^{-1} = \omega(e^{-\theta} k)$. Since $\omega_\theta = e^{-\theta}\omega$, we have $H_f(\theta) = e^{-\theta} H_f$.

We do not dilate the particle coordinates, and hence we lift the dilatation $U_f(\theta)$ to $\mathscr{H} = \mathscr{H}_{el} \otimes \mathscr{F}$ by

$$U(\theta) = \mathbf{1}_{el} \otimes U_f(\theta). \tag{I.64}$$

Remembering the definition of H_g, we obtain

$$H_g(\theta) = H_{el} \otimes \mathbf{1}_f + e^{-\theta} \mathbf{1}_{el} \otimes H_f + W_g^{(\theta)},$$

where $W_g^{(\theta)} := U(\theta)\, W_g\, U(\theta)^{-1}$. Moreover, $W^g_{(\theta)} = g W_1^{(\theta)} + g^2 W_2^{(\theta)}$, where $W_l^{(\theta)} := U(\theta)\, W_l U(\theta)^{-1}$. Applying $U(\theta)$ to W_1 and W_2 by using (I.63), one easily checks that $W_l^{(\theta)}$ is obtained from W_l by the replacement

$$G_{m,n}(\underline{k}) \to G^{(\theta)}_{m,n}(\underline{k}) := e^{(3\theta/2)(m+n)}\, G_{m,n}(e^{-\theta}\underline{k}) \tag{I.65}$$

in (I.26) and (I.27), where $\underline{k} \in \mathbb{R}^{3(m+n)}$. By Hypothesis 2, $\theta \mapsto H_g(\theta)$ defines an analytic family of type A.

We note that the dilatation $U(\theta)$ above dilates the photon momenta but leaves the particle coordinates unchanged. This has the strange effect that terms like $\exp(ik \cdot x)$ are transformed into $\exp(ie^{-\theta} k \cdot x)$ which grow exponentially in some directions in X, unless a spatial cutoff in the particle coordinates that decays super-exponentially fast is imposed. We come back to this point in Section I.6.

We conclude our discussion of Theorem I.9 by showing how to analytically continue matrix elements of the resolvent of H_g across the spectrum of H_g. We obtain this analytic continuation by the Combes argument. Namely, by the unitarity of $U(\theta)$ for real θ,

$$\langle u|\, (H_g - z)^{-1}\, v\rangle = \langle u(\bar{\theta})|\, (H_g(\theta) - z)^{-1}\, v(\theta)\rangle, \tag{I.66}$$

where $u(\theta) = U(\theta) u$, etc., for $z \in \mathbb{C}_+$. Assume now that $u(\bar{\theta})$ and $v(\theta)$ have analytic continuations into a complex neighbourhood of $\theta = 0$. Then the r.h.s. of (I.66) has an analytic continuation in θ into a complex neighbourhood of $\theta=0$. Since Eq. (I.66) holds for real θ, it also holds in the above neighbourhood. Fix θ on the r.h.s. of (I.66), with $\operatorname{Im}\theta>0$. The r.h.s. of (I.66) can be analytically continued across the real axis into the part of the resolvent set of $H_g(\theta)$ which lies in $\overline{\mathbb{C}_-}$. This yields an analytic continuation of the l.h.s. of (I.66). We show in Chapter V that the thresholds of $H_g(\theta)$, i.e., the points at which branches of continuous spectrum of $H_g(\theta)$ originate, and the eigenvalues of $H_g(\theta)$ are *independent* of θ. Thus the (complex) eigenvalues of $H_g(\theta)$ are the resonances of H_g. In fact, the justification of this statement is not easy, since, as we will see later, the eigenvalues in question lie at the tips of continuous spectrum of $H_g(\theta)$, and therefore adequate control of the latter is needed. See Chapters IV–V and [5].

We have assumed above that u and v are such that $u(\bar{\theta})\equiv U(\bar{\theta})u$ and $v(\theta)\equiv U(\theta)v$ have analytic continuations into a complex neighbourhood of $\theta=0$. The set of such vectors is dense in $\mathscr{H}_{el}\otimes\mathscr{F}$. However, it is not the set $\mathscr{D}$. In order for $u(\bar{\theta})$, with $u\in\mathscr{D}$, to have an analytic continuation into a neighbourhood of $\theta=0$, we have to modify the deformation family $U(\theta)$. We briefly sketch such a modification. Let $u\in\mathscr{D}$, and let V be a vector field on $\mathbb{R}^3$ supported outside the support of the one-photon part of u and approaching the identity vector field, $V_0(k)=k$, as $|k|\to\infty$. Define ϕ_θ to be either the flow generated by V (see [38]) or the shift along V, i.e., $\phi_\theta(k)=k+\theta V(k)$, (a linear approximation to the flow above, see [9, 21]). Both definitions have their advantages. We shall appeal to the second one, since it obviously guarantees the analyticity in θ needed for our purposes. Now we define U_θ on the one-photon space by

$$U_\theta\colon u\to\sqrt{\mathrm{Jac}}\,u\circ\phi_\theta, \tag{I.67}$$

where Jac is the Jacobian of the transformation $k\to\phi_\theta(k)$. Using (I.63), we lift U_θ to $\mathscr{H}_f=\mathscr{F}$ and then, using (I.64), to $\mathscr{H}_{el}\otimes\mathscr{H}_f$. The resulting transformation $U(\theta)$ has the desired property that $U(\theta)u$ has an analytic continuation into a neighbourhood of $\theta=0$, for $u, v\in\mathscr{D}$.

Note that the Riemann surface of $(H_g-z)^{-1}$ is *independent* of the choice of the transformation $U(\theta)$, while the cuts depend only on the behaviour of $V(k)$ at infinity.

Now we define the thresholds of $H_g(\theta)$ as fixed points of $\sigma(H_g(\theta))$ under small variations of θ (i.e., as branch point of the Riemann surface of H_g). It turns out that the eigenvalues of $H_g(\theta)$ are either isolated or located at its thresholds, and thus they are independent of θ.

The real eigenvalues of $H(\theta)$, for $\operatorname{Im}\theta>0$, are just the eigenvalues of H_g, while the complex eigenvalues are located at the singularities of the analytic continuation of $z\mapsto\langle u,(H_g-z)^{-1}v\rangle$, with $u, v\in\mathscr{D}$, onto the second Riemann sheet and therefore are identified with the resonances of H_g.

Thus, to find resonances of H_g, we have to locate complex eigenvalues of $H_g(\theta)$ for an appropriate θ with $\operatorname{Im}\theta > 0$.

Observe that the eigenvalues of $H_0(\theta)$ are real and, by separation of variables, the fact that $\sigma_{\mathrm{pp}}[H_f(\theta)] = \{0\}$, and $\sigma_{\mathrm{cont}}[H_f(\theta)] = e^{-\theta}\mathbb{R}_+$, we have

$$\sigma_{\mathrm{pp}}[H_0(\theta)] = \sigma_{\mathrm{pp}}[H_{el}] \qquad \text{and} \qquad \sigma_{\mathrm{cont}}[H_0(\theta)] = \sigma[H_{el}] + e^{-\theta}\overline{\mathbb{R}_+}.$$

I.6. *Comparison to the Standard Model*

We come back to our discussion at the end of Section I.1: the coupling functions $\mathbb{G}_{m,n}$ in (I.16)–(I.18) contain terms growing linearly in $\mathbf{x}_j$. Thus, $\mathbb{H}_2$ fails to be a relatively bounded perturbation of $\mathbb{H}_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f$, unless we impose a spatial cutoff in the particle coordinates or assume the potential $\mathbb{V}$ to be confining. Thus, together with the help of the UV-cutoff κ that regularizes the electromagnetic vector potential, $\mathbf{A} \to \mathbf{A}_{\bar{\kappa}}$, we need to restrict the electron coordinates to small distances from the nucleus. To this end, we have introduced an analytic function $\bar{\kappa}$ obeying $\bar{\kappa}(0) = 1$ and decaying sufficiently fast at infinity. To be specific, we choose $\bar{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^4)$. So, $\bar{\kappa}(\mathrm{e}^{-\theta}\mathbf{k}/K)$ acts as an ultraviolet cutoff, cutting off the photon momenta on scale K (keeping θ fixed). Similarly, we replace

$$\mathbb{G}_{m,n} \to \mathbb{G}_{m,n;\,\mathrm{reg}} := \prod_{j=1}^{N} \bar{\kappa}(\mathbf{x}_j / r_{el})\, \mathbb{G}_{m,n} \tag{I.68}$$

in (I.16)–(I.18). Here, $\bar{\kappa}(\mathbf{x}/r_{el})$ imposes a spatial cutoff on the electron variables x at length scale r_{el}. In physics, one would choose r_{el} of the order $\max_n Z_n^{-1}$, the diameter of the atom or molecule under consideration.

As we have pointed out in the preceeding section, if we ignored the spatial cutoff functions $\bar{\kappa}(\mathbf{x}/r_{el})$ then $\mathbb{G}^{\theta}_{m,n;\,\mathrm{reg}}$ would grow exponentially as $|k| \to \infty$. One way of avoiding this growth is to simultaneously dilate the electron coordinates $\mathbf{x} \mapsto \mathrm{e}^{\theta}\mathbf{x}$ and the photon variable $\mathbf{k} \mapsto \mathrm{e}^{-\theta}\mathbf{k}$. In order not to unnecessarily encumber the present exposition, we refrain from doing so and, rather, damp $\|\mathbb{G}^{\theta}_{m,n;\,\mathrm{reg}}\|$ by a rapidly decaying cutoff, $\bar{\kappa}$.

Now, we will show that $\mathbb{G}^{\theta}_{m,n;\,\mathrm{reg}}$ fulfills Hypotheses 2 and 3, provided we choose $\bar{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^4)$, for example. We remark that the choice $\tilde{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^2)$ would not suffice at this point because (I.71) would not hold—unless we additionally assume $gKr_{el} \ll 1$.

THEOREM I.10. *Let $\bar{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^4)$. Then $\mathbb{G}^{(\theta)}_{m,n;\,\mathrm{reg}}$ fulfills Hypothesis 2 and 3 with*

$$J(\mathbf{k}) \leqslant C_0\, |\mathbf{k}|^{1/2} \exp[-|\mathbf{k}|^4 / C_0], \tag{I.69}$$

for some constant C_0 depending on $\mathbb{V}$ and $\vartheta_0 > 0$, where $|\operatorname{Im}\theta| \leqslant \vartheta_0$.

Proof. We will give the proof only for $\mathbb{G}^{(\theta)}_{1,0;\,\mathrm{reg}}$ and $\mathbb{G}^{(\theta)}_{2,0;\,\mathrm{reg}}$; but the other cases are similar. For our argument we need three basic estimates

$$|\exp[-\mathrm{e}^{-4\theta}|\mathbf{k}|^4/K^4 - |\mathbf{x}|^4/r_{el}^4]|\,|\mathbf{x}|$$
$$\leqslant \mathrm{const}\cdot r_{el}\cdot \exp[-\mathrm{e}^{\pi}|\mathbf{k}|^4/K^4], \tag{I.70}$$
$$|\exp[-\mathrm{e}^{-4\theta}|\mathbf{k}|^4/K^4 - |\mathbf{x}|^4/r_{el}^4 + \alpha \mathrm{e}^{|\theta|}|\mathbf{k}|\,|\mathbf{x}|]|$$
$$\leqslant \mathrm{const}\cdot r_{el}\cdot \exp[-\mathrm{e}^{-\pi}|\mathbf{k}|^4/(2K^4) - |\mathbf{x}|^4/(2r_{el}^4)], \tag{I.71}$$
$$|\exp[-i\alpha \mathrm{e}^{-\theta}\mathbf{k}\cdot\mathbf{x}] - 1|$$
$$\leqslant \alpha\cdot \mathrm{e}^{|\theta|}|\mathbf{k}|\cdot|\mathbf{x}|\exp[\alpha \mathrm{e}^{|\theta|}|\mathbf{k}|\,|\mathbf{x}|], \tag{I.72}$$

the proofs of which are elementary. Since $\mathbb{G}^{(\theta)}_{2,0;\,\mathrm{reg}}(\mathbf{k},\lambda;\mathbf{k}',\lambda')$ is a multiplication operator on $\mathscr{H}_{el}$, its norm is given by the supremum of its modulus. Thus

$$\|\mathbb{G}^{(\theta)}_{2,0;\,\mathrm{reg}}(\mathbf{k},\lambda;\mathbf{k}',\lambda')\| \leqslant \frac{8N}{\pi}\sup_{\mathbf{x}\in\mathbb{R}^3}\{f(\mathbf{k},\mathbf{x})\,f(\mathbf{k}',\mathbf{x})\}, \tag{I.73}$$

where

$$f(\mathbf{k},\mathbf{x}) := \omega(\mathbf{k})^{-1/2}\,|\exp[-\mathrm{e}^{-4\theta}|\mathbf{k}|^4/K^4 - |\mathbf{x}|^4/r_{el}^4]|$$
$$\cdot|\exp[-i\alpha \mathrm{e}^{-\theta}(\mathbf{k}\cdot\mathbf{x})] - 1|$$
$$\leqslant \mathrm{const}\;\alpha r_{el}\mathrm{e}^{\pi/4}\omega(\mathbf{k})^{1/2}\exp[-e^{\pi}|\mathbf{k}|^4/(2K^4)], \tag{I.74}$$

successively applying (I.70), (I.71), and (I.72). The right side of (I.74) yields the desired estimate for $\mathbb{G}^{(\theta)}_{m,n;\,\mathrm{reg}}$.

Returning to the definition of $\mathbb{G}^{(\theta)}_{1,0;\,\mathrm{reg}}(\mathbf{k},\lambda)$, we observe that it is a sum of three terms corresponding to the r.s. in (I.16). Two of these are multiplication operators on $\mathscr{H}_{el}$, and they are shown to be bounded in the same way as we have just demonstrated for $\mathbb{G}^{(\theta)}_{2,0;\,\mathrm{reg}}(\mathbf{k},\lambda;\mathbf{k}',\tau')$. So, all terms in $\mathbb{G}^{(\theta)}_{m,n;\,\mathrm{reg}}$ are bounded operators, except one which is only relatively $(\sum_{n=1}^{N} -\Delta_n)^{1/2}$-bounded. But this is sufficient for the estimate (I.37). ∎

II. BINDING

In this chapter we address the issue of binding, i.e., Problem (a) described in Section I.2. Theorem II.8 or, equivalently, Theorem I.1(a) represent our solution of this problem. Our approach is based on introducing an IR (infrared) cutoff, obtaining estimates on the cutoff problem that are *uniform* in the cutoff and then removing the cutoff and showing that the cutoff objects converge to those of the original problem. Note that the role of the IR cutoff is to make the eigenvalues of interest isolated. The manner in which we realize the cutoff is not very important. We do it by decoupling photons of energies $\leqslant m$ from the electrons, so our estimates must be uniform in m. In this chapter, we require Hypothesis 1, i.e., we consider the operator $H_g = H_0 + gW_x$ with the assumption that (I.33)–(I.34) hold. To ease the handling of the many sub- and superindices that we introduce below, we write $H \equiv H_g$ in this section.

II.1. *Exponential Bounds*

THEOREM II.1. *Let* $\Delta \subset \mathbb{R}$ *be an interval with* $\sup \Delta < \Sigma := \inf \sigma_{\text{cont}}(H_{el}) \leqslant 0$ *and let* $\alpha, g > 0$ *satisfy* $\alpha^2 < \Sigma - \sup \Delta - g^2 \sup_x \int |G_x|^2/\omega$. *Then there exists a constant,* $M_\Delta < \infty$, *depending only on* V, Λ_1, *and* Δ, *such that*

$$\|e^{\alpha |x|} \otimes \mathbf{1}_f \chi_\Delta(H)\| \leqslant M_\Delta < \infty. \tag{II.1}$$

Proof. The idea of the proof is based on the fact (see e.g. [10, 23]) that, for any $\varepsilon > 0$, there is $R > 0$ such that

$$H_{el} \geqslant \Sigma - \varepsilon \qquad \text{on } H^1(\{|x| \geqslant R\}). \tag{II.2}$$

Let χ_R be the characteristic function of the set $\{|x| \geqslant R\}$. Denote by H_R the operator obtained from H by replacing the particle potential $V(x)$ by $V(x)\,\chi_R(x)$. Then, by the inequalities $\pm gW_x \leqslant \varepsilon H_f + (g^2\Lambda^2/\varepsilon)$, where $\Lambda = \sup_x\{\int |G_x|^2/\omega\}^{1/2}$, and by an analogue of (II.2), we have that

$$H_R \geqslant [\tfrac{1}{2} H_f - 2g^2\Lambda^2 + \Sigma - \varepsilon].$$

Hence by the condition on Δ, and for g and ε sufficiently small, $\sup \Delta < \Sigma - \varepsilon - 2g^2\Lambda^2$, and therefore

$$\chi_\Delta(H_R) = 0.$$

This implies that

$$\chi_\Delta(H) = \chi_\Delta(H) - \chi_\Delta(H_R). \tag{II.3}$$

Without loss of generality we can replace $\chi_\Delta(\mu)$ by a smooth function which we again denote by $\chi_\Delta(\mu)$. In this case there is a smooth function f on $\mathbb{C}$ which is almost analytic [31] (i.e., $\partial_{\bar z} f = 0$ on $\mathbb{R}$) and compactly supported, such that

$$\chi_\Delta(A) = \int df(z)(z - A)^{-1}, \tag{II.4}$$

where $df(z) = -(1/2\pi)\,\partial_{\bar z} f(z)\,dx\,dy$, $z = x + iy$, for any selfadjoint operator A. Moreover, the support of f can be taken to be inside an arbitrarily small complex neighbourhood of the interval Δ (see [23]). Equation (II.3) then implies

$$\chi_\Delta(H) = \int df(z)[(z-H)^{-1} - (z-H_R)^{-1}]$$

$$= -\int df(z)(z-H_R)^{-1}\,\overline{\chi_R}\,V(z-H)^{-1},$$

where $\overline{\chi_R} = \mathbf{1} - \chi_R$. Now, let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth, convex function with $\phi \equiv 0$ on $(0, 2]$ and $\phi(r) = r - 3$ for $r \geqslant 4$. Then $|\phi'| \leqslant 1$. Defining $F(x) := \alpha R\phi(|x|/R)$, we observe that $F \equiv 0$ on $\{|x| \leqslant 2R\}$ and $F(x) = \alpha\,|x| - 3\alpha$ for all $|x| \geqslant 4R$. Moreover, $|\nabla_x F| \leqslant \alpha$. Define $H_{R,F} = \mathrm{e}^F H_R \mathrm{e}^{-F}$. Then

$$\mathrm{e}^F(z-H_R)^{-1}\,e^{-F} = (z-H_{R,F})^{-1}.$$

Since F is chosen such that $\mathrm{e}^F\overline{\chi_R} = \overline{\chi_R}$, the last two relations yield

$$\mathrm{e}^F\chi_\Delta(H) = -\int df(z)(z-H_{R,F})^{-1}\,\overline{\chi_R}\,V(z-H)^{-1}. \tag{II.5}$$

In order to estimate the first factor on the r.h.s., we write

$$H_{R,F} = H_R - |\nabla_x F|^2 - iA_F,$$

where $A_F = \frac{1}{2}(p\cdot\nabla_x F + \nabla_x F\cdot p)$, with $p = -i\,\nabla_x$. Having chosen supp $f \ni z$ sufficiently close to Δ and ε sufficiently small, we note that $H_R - |\nabla_x F|^2 - \mathrm{Re}(z) \geqslant \Sigma - \varepsilon - 2g^2\Lambda^2 - \alpha^2 - \mathrm{Re}(z) \geqslant \mathrm{const} > 0$. Moreover, A_F is a perturbation of $Q := H_R - |\nabla_x F|^2 - \mathrm{Re}(z)$ of relative bound zero. Thus

$$\|(H_{R,F} - z)^{-1}\| \leqslant \|Q^{-1/2}\|^2 \cdot \|(\mathbf{1} - iQ^{-1/2}(A_F + \mathrm{Im}\,z)\,Q^{-1/2})^{-1}\|$$

$$\leqslant \|(H_R - |\nabla_x F|^2 - \mathrm{Re}\,z)^{-1}\| \leqslant \mathrm{const}.$$

This, together with (II.5), yields

$$\|\mathrm{e}^F \otimes \mathbf{1}_f\,\chi_\Delta(H)\| \leqslant \mathrm{const}\,\|V(H+i)^{-1}\| \cdot \int \left\|\frac{H+i}{H-z}\right\| |df(z)|.$$

Since $|\partial_{\bar z} f(z)| \leqslant \mathrm{const}\cdot|y|$, with $z = x + iy$, and f is compactly supported, the r.h.s. is bounded. ∎

II. 2. *Existence of the Ground State for H_m, $m>0$*

We begin by introducing an IR cutoff in the interaction, replacing $W \equiv W_x = \int [G_x(k) \otimes a^\dagger(k) + \bar{G}_x(k) \otimes a(k)]\, dk$ by

$$W_m = \int_{\{\omega(k) \geqslant m\}} [G_x(k) \otimes a^\dagger(k) + \bar{G}_x(k) \otimes a(k)]\, dk. \tag{II.6}$$

We define

$$H_m = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + g W_m = H_0 + g W_m \tag{II.7}$$

Our task is to prove that H_m has a ground state and then to obtain good control of the latter.

THEOREM II.2. *Let $\Sigma := \inf \sigma_{cont}(H_{el})$, $E_0 := \inf \sigma(H_{el})$ and assume that*

$$\Sigma - E_0 - g^2 \sup_x \left\{ \int |G_x(k)|^2\, \omega^{-1}(k)\, dk \right\} \geqslant 4m > 0. \tag{II.8}$$

Then H_m has a ground state ϕ_m at the bottom $E_m := \inf \sigma(H_m)$ of its spectrum. Moreover, ϕ_m is unique, provided that $G_x(k) = \overline{G_x(-k)}$, for all $k \in \mathbb{R}^3$.

Proof. We begin the proof with the remark that if the one-particle Hilbert space h is a direct sum $h_1 \oplus h_2$, then the bosonic Hilbert space $\mathscr{F}$ over h is isomorphic to $\mathscr{F}_1 \otimes \mathscr{F}_2$, where $\mathscr{F}_i$ is the Fock space over h_i. Indeed, if $\{f_j\}$ is an orthonormal basis in h_1 and $\{g_j\}$ is an orthonormal basis in h_2, this isomorphism is given by

$$\begin{aligned} &a^\dagger(f_1) \cdots a^\dagger(f_k)\, a^\dagger(g_1) \cdots a^\dagger(g_l) \Omega \\ &\qquad \mapsto a^\dagger(f_1) \cdots a^\dagger(f_k)\, \Omega_1 \otimes a^\dagger(g_1) \cdots a^\dagger(g_l) \Omega_2. \end{aligned}$$

In the present situation we have $h = L^2(\mathbb{R}^3) = L^2(\mathscr{A}_l) \oplus L^2(\mathscr{A}_s)$, where $\mathscr{A}_s := \{k \mid \omega(k) < m\}$ and $\mathscr{A}_l := \mathbb{R}^3 \backslash \mathscr{A}_s$. The isomorphism above maps H_f to $H_{f,l} \otimes \mathbf{1}_s + \mathbf{1}_l \otimes H_{f,s}$ and $g W_m$ to $g W_m \otimes \mathbf{1}_s$. Here, $H_{f,s/l} := \int_{\mathscr{A}_{s/l}} \omega a^\dagger a$. In this representation, H_m appears as

$$H_m = H_{m,l} \otimes \mathbf{1}_s + (\mathbf{1}_{el} \otimes \mathbf{1}_l) \otimes H_{f,s}, \tag{II.9}$$

as an operator on $(\mathscr{H}_{el} \otimes \mathscr{F}_l) \otimes \mathscr{F}_s$, where

$$H_{m,l} = H_{el} \otimes \mathbf{1}_l + \mathbf{1}_{el} \otimes H_{f,l} + g W_m.$$

From Representation (II.9) it is obvious that H_m has a ground state $\phi_m \in (\mathscr{H}_{el} \otimes \mathscr{F}_l) \otimes \mathscr{F}_s$ if and only if $H_{m,l}$ has a ground state $\phi_{m,l} \in \mathscr{H}_{el} \otimes \mathscr{F}_l$.

Indeed, in this case $\phi_m = \phi_{m,l} \otimes \Omega_s$. Thus the existence of the ground state ϕ_m follows from Lemma II.3 below. We postpone the uniqueness part of the proof after the proof of Lemma II.3.

LEMMA II.3. *Under the conditions of Theorem* II.2 $H_{m,l}$ *has a ground state* $\phi_{m,l}$ *at the bottom* $E_m := \inf \sigma(H_m) = \inf \sigma(H_{m,l})$ *of its spectrum.*

Proof. The claim follows by proving that, for any $0 < m_1 < m$,

$$\dim \operatorname{Ran}\{\chi_{H_{m,l} \leqslant E_m + m_1}\} < \infty. \tag{II.10}$$

To prove this inequality, we employ a discretization, i.e., a family of operators $H^{\varepsilon}_{m,l}$, for $\varepsilon > 0$, with $\|(H^{\varepsilon}_{m,l} - H_{m,l})(H_{m,l} - E_m + 1)^{-1}\| \to 0$, as $\varepsilon \to 0$. This ensures that $H_{m,l} - E_m - m_1 \leqslant H^{\varepsilon}_{m,l} - E_m - m_2$, for any $m_1 < m_2 < m$, and hence,

$$\dim \operatorname{Ran}\{\chi_{H_{m,l} \leqslant E_m + m_1}\} = \dim \operatorname{Ran}\{\chi_{H^{\varepsilon}_{m,l} \leqslant E_m + m_2}\}$$

provided $\varepsilon > 0$ is sufficiently small. Therefore, (II.10) holds, provided we prove that $\dim \operatorname{Ran}\{\chi_{H^{\varepsilon}_{m,l} \leqslant E_m + m}\} < \infty$, for sufficiently small $\varepsilon > 0$, which we demonstrate below.

We now construct the discretized operators $H^{\varepsilon}_{m,l}$. Consider $\varepsilon\omega(k)$ as a scale function on $\mathbb{R}^3$ and partition $\mathbb{R}^3$ w.r. to this function. Namely, we decompose $\mathbb{R}^3$ into a disjoint union of "cubes" $C_{\alpha,\varepsilon}$ s.t.

$$c\varepsilon \operatorname{dist}(C_{\alpha,\varepsilon}, 0) \leqslant \operatorname{diam}(C_{\alpha,\varepsilon}) \leqslant C\varepsilon \operatorname{dist}(C_{\alpha,\varepsilon}, 0)$$

for some universal c and C (1 and 4, say). One can find such a decomposition by breaking $\mathbb{R}^3$ first into the shells $\{\varepsilon 2^{-n-1} \leqslant |k| < \varepsilon 2^{-n}\}$, $n \in \mathbb{Z}$, and then breaking each shell into segments by intersecting it with cones based on a partition of the unit sphere, S^2, into sets of equal size. (Alternatively, we observe that $\varepsilon\,|k|$ is a distance of k to the set $\{0\}$ and appeal to a general result in [42] (Chapter VI, Theorem 1), giving the desired decomposition into cubes, indeed. Note, in parentheses, that such a decomposition is locally finite, i.e., every point of $\mathbb{R}^3$ belongs to closures of a uniformly bounded number of cubes.) Define $\omega_\varepsilon(k) = \operatorname{Av}_{C_{\alpha,\varepsilon}}(\omega) \equiv (\operatorname{vol} C_{\alpha,\varepsilon})^{-1} \int_{C_{\alpha,\varepsilon}} \omega$, where $C_{\alpha,\varepsilon} \ni k$, and

$$H^{\varepsilon}_{f,l} = \int_{\mathscr{A}_l} \omega_\varepsilon a^\dagger a$$

Note that by the choice of the $C_{\alpha,\varepsilon}$'s

$$\sup \frac{|\omega - \omega_\varepsilon|}{\omega} \leqslant \text{const} \cdot \varepsilon \tag{II.11}$$

and therefore

$$\pm(H_{f,l} - H^{\varepsilon}_{f,l}) \leqslant \text{const} \cdot \varepsilon H_{f,l} \tag{II.12}$$

Similarly, we define gW^{ε}_m by replacing $G_x(k)$ by $G^{\varepsilon}_x := \mathrm{Av}_{C_{\alpha,\varepsilon}}(G_x)$ for $C_{\alpha,\varepsilon} \ni k$. We set

$$H^{\varepsilon}_{m,l} = H_{el} \otimes \mathbf{1}_l + \mathbf{1}_{el} \otimes H^{\varepsilon}_{f,l} + gW^{\varepsilon}_m. \tag{II.13}$$

Now we show that $\|(H^{\varepsilon}_{m,l} - H_{m,l})(H_{m,l} - E_m + 1)^{-1}\| \to 0$, as $\varepsilon \to 0$.

The second resolvent equation, Corollary I.7 and Eq. (II.12) imply that

$$\begin{aligned} &\|(H^{\varepsilon}_{m,l} - H_{m,l})(H_{m,l} - E_m + 1)^{-1}\| \\ &\quad \leqslant \text{const} \left\{ \varepsilon + \sup_x \int (1 + \omega^{-1}(k)) \, |G_{x,\varepsilon}(k) - G_x(k)|^2 \, dk \right\}, \end{aligned}$$

and the right side converges to 0, as $\varepsilon \to 0$, by Lebesgue dominated convergence.

We complete the proof by showing that $\dim \mathrm{Ran}\{\chi_{H^{\varepsilon}_{m,l} \leqslant E_m + m_2}\} < \infty$ for small but non-zero $\varepsilon > 0$, i.e., that $H^{\varepsilon}_{m,l} - E_m - m_2$ has only finitely many negative eigenvalues. To this end, we decompose the one-photon state space into discrete and fluctuating parts as

$$L^2(\mathscr{A}_l, dk) = h^d \oplus h^f, \tag{II.14}$$

where h^d (the "discrete subspace") is the l^2-space of functions which are constant on the cubes $C_{\alpha,\varepsilon}$, while h^f (the "fluctuating subspace") is the orthogonal complement of h^d. As in the proof of Theorem II.2, the decomposition (II.14) leads to the representation

$$\mathscr{H}_{el} \otimes \mathscr{F}_l = (\mathscr{H}_{el} \otimes \mathscr{F}_d) \otimes \mathscr{F}_f,$$

on which $H^{\varepsilon}_{m,l}$ decomposes as

$$H^{\varepsilon}_{m,l} = H^{\varepsilon}_{m,d} \otimes \mathbf{1}_f + \mathbf{1}_d \otimes H^{\varepsilon}_{f,f} \tag{II.15}$$

where

$$H^{\varepsilon}_{m,d} = H_{el} \otimes \mathbf{1}_d + \mathbf{1}_{el} \otimes H^{\varepsilon}_{f,d} + gW^{\varepsilon}_m. \tag{II.16}$$

Note that

$$E_m^\varepsilon := \inf \sigma(H_{m,l}^\varepsilon) = \inf \sigma(H_{m,d}^\varepsilon) \leqslant E_0.$$

Let us denote the projection onto the vacuum in $\mathscr{F}_{d/f}$ by $P_{d/f}$, remarking that $H_{f,f}^\varepsilon P_f^\perp \geqslant m P_f^\perp$. Hence,

$$\begin{aligned} H_{m,l}^\varepsilon - E_m^\varepsilon - m &\geqslant (H_{m,d}^\varepsilon - E_m^\varepsilon) \otimes P_f^\perp + (H_{m,d}^\varepsilon - E_m^\varepsilon - m) \otimes P_f \\ &\geqslant (H_{m,d}^\varepsilon - E_m^\varepsilon - m) \otimes P_f. \end{aligned} \tag{II.17}$$

Next, we set $P_{el} := \chi_{H_{el} \leqslant \Sigma - m}$, so that $H_{el} \geqslant E_0 P_{el} + (\Sigma - m) P_{el}^\perp$. Note that P_{el} is finite dimensional. We use (I.60) which implies that for any $\lambda > 1$, we have

$$g W_m^\varepsilon > -\lambda^{-1} \mathbf{1}_{el} \otimes H_{f,d}^\varepsilon - \lambda F^\varepsilon, \tag{II.18}$$

where $F^\varepsilon := \sup_x \int_{\mathscr{A}_l} |G_x^\varepsilon(k)|^2 \, \omega_\varepsilon^{-1} \, dk$. Passing to sufficiently small $\varepsilon > 0$, we may assume that $F^\varepsilon \leqslant m + \sup_x \int |G_x(k)|^2 \, \omega^{-1}(k) \, dk \leqslant \Sigma - E_0 - 3m$, according to (II.8). Thus

$$\begin{aligned} H_{m,d}^\varepsilon &\geqslant H_{el} \otimes \mathbf{1}_d - \lambda F^\varepsilon + (1 - \lambda^{-1}) \, \mathbf{1}_{el} \otimes H_{f,d}^\varepsilon \\ &\geqslant P_{el}^\perp \otimes (\Sigma - m - \lambda F^\varepsilon) + P_{el} \otimes ((1 - \lambda^{-1}) \, H_{f,d}^\varepsilon + E_0 - \lambda F^\varepsilon). \end{aligned} \tag{II.19}$$

Choosing $\lambda := 1 + m(\Sigma - E_0)^{-1}$, and remembering that $E_0 \geqslant E_m^\varepsilon$, we obtain

$$H_{m,l}^\varepsilon - E_m - m_2 \geqslant H_{m,l}^\varepsilon - E_m^\varepsilon - m \geqslant P_{el} \otimes ((1 - \lambda^{-1}) \, H_{l,d}^\varepsilon - \Sigma + E_0) \otimes P_l, \tag{II.20}$$

for $\varepsilon > 0$ sufficiently small.

We conclude the proof by using the min-max principle, which implies, together with (II.20), that

$$\dim \mathrm{Ran}\{\chi_{H_{m,l}^\varepsilon \leqslant E_m + m_2}\} \leqslant \dim \mathrm{Ran}\{P_{el}\} \cdot \dim \mathrm{Ran}\{\chi_{H_{f,d}^\varepsilon \leqslant m^{-1}(\Sigma - E_0 + m)^2}\}. \tag{II.21}$$

The right side is clearly finite, since $H_{f,d}^\varepsilon$ has discrete spectrum of finite multiplicity in any interval $[0, a]$. ∎

II.3. *Uniqueness of the Ground State of H_m, for $m > 0$*

To prove uniqueness of the ground state in the case where $\mathscr{H}_{\mathrm{at}} = L^2(\mathbb{R}^{3N})$ (ignoring the statistics of electrons) and $G_x(k) = \overline{G_x(-k)}$, we use a Perron–Frobenius argument (see [16]). We consider the Schrödinger representation, $L^2(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C)$, of the Fock space $\mathscr{F}$, where $d\mu_C$ is the Gaussian measure with mean 0 and covariance operator $C = (2\omega)^{-1}$. In the Schrödinger representation, $W = a(G_x) + a^\dagger(G_x)$ is a *real* multiplication operator. Furthermore, as follows from [16], the operators $\exp(-tH_f)$ are positivity preserving and ergodic, for all $t>0$. Ergodicity means that, for arbitrary non-negative functions ψ and ϕ in $L^2(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C)$ of positive L^2-norm,

$$\langle \psi \,|\, \mathrm{e}^{-tH_f}\phi \rangle > 0,$$

for t large enough. The proof of ergodicity presented in [16] does not quite cover our case, but a variant of their proof still holds: First, note that the vacuum Ω in $L^2(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C)$ is the functional identically equal to 1, $\Omega \equiv 1$. Let P_Ω be the orthogonal projection onto Ω, and $P_\Omega^\perp = \mathbf{1} - P_\Omega$. Then

$$\begin{aligned} \langle \psi \,|\, \mathrm{e}^{-tH_f}\phi \rangle &= \langle \psi \,|\, P_\Omega \phi \rangle - \langle \psi, \mathrm{e}^{-tH_f} P_\Omega^\perp \phi \rangle \\ &\geqslant \int \psi \int \phi - \|\mathrm{e}^{-t/2H_f} P_\Omega^\perp \psi\| \; \|\mathrm{e}^{-t/2H_f} P_\Omega^\perp \phi\|, \end{aligned}$$

where $\int \psi \equiv \int_{\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^3)} \psi(\varphi)\, d\mu_C(\varphi)$. Since ψ and ϕ are non-negative functions, there exists some $\delta > 0$, depending on ψ and ϕ, such that

$$\int \psi \int \phi \geqslant \delta^2 \, \|\psi\| \; \|\phi\|$$

Since the spectrum of H_f is absolutely continuous on Ran $P_\Omega^\perp$, there is some $\varepsilon > 0$ (depending on ψ and ϕ) such that

$$\|\chi_{H_f \leqslant \varepsilon} P_\Omega^\perp \psi\| \leqslant \frac{\delta}{4} \|\psi\|,$$

and similarly for ϕ.

Next, we choose t_0 so large that

$$\|\mathrm{e}^{-(t/2)\, H_f} \chi_{H_f \geqslant \varepsilon} \psi\| \leqslant \mathrm{e}^{-t\varepsilon/2} \, \|\psi\| \leqslant \frac{\delta}{4} \, \|\psi\|,$$

for arbitrary $t \geqslant t_0$, and similary for ϕ. Combining all these inequalities, we conclude that

$$\begin{aligned} \langle \psi \,|\, e^{-tH_f}\phi \rangle &\geqslant \int \psi \int \phi - \|\mathrm{e}^{-(t/2)\, H_f} P_\Omega^\perp \psi\| \cdot \|\mathrm{e}^{-(t/2)\, H_f} P_\Omega^\perp \phi\| \\ &\geqslant \tfrac{3}{4} \delta^2 \, \|\psi\| \cdot \|\phi\| > 0, \end{aligned}$$

for $t \geqslant t_0$. This proves ergodicity even for the *massless*, scalar field.

The theory of Schrödinger operators tells us that, under our assumptions on $V(x)$, $e^{-tH_{el}}$, for $t>0$, is a positivity preserving and ergodic operator on $L^2(\mathbb{R}^{3N}))$; (see e.g. Theorems XIII.44 and XIII.46 in [37]). Hence e^{-tH_0}, with $H_0 = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f$, is positivity preserving and ergodic on $L^2(\mathbb{R}^{3N}) \otimes L^2(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C)$, for $t>0$. Define the bounded real multiplication operator $W_{m,N} = W_m \chi_{|W_m| \leqslant N}$. Since $W_{m,N}$ is H-bounded, we conclude that

$H_0 + W_{m,N}$ converges to $H_0 + W_m \equiv H_m$ in the strong resolvent sense, as $N \to \infty$. Hence, by Theorems XIII.43 and XIII.45 in [37], e^{-tH_m} is positivity preserving and ergodic. The last property implies that the ground state is strictly positive and therefore unique (see Theorem XIII.43 in [37]). This finishes the proof of Theorem II.2. ∎

Recall that $(\Lambda_1)^2 = \sup_x \int [1 + \omega(k)^{-1}] \, |G_x|^2 \, dk < \infty$. Thus, due to Corollary I.7, H_m is resolvent norm continuous in g. We have shown that, for each g (including $g = 0$), the interval of length $m/2$ at the bottom of the spectrum of H_m consists of a finite number of eigenvalues of finite multiplicity. General perturbation theory implies that those eigenvalues are continuous in g.

Finally, we point out that W_m and thus H_m are subject to Hypothesis 1. In particular, V, Σ, Λ_1, and $\Lambda_{2,\varepsilon}$ are the same as for H. Consequently, the exponential decay estimate (II.1) holds for H_m, as it does for H, with $M_\Delta < \infty$ uniformly in $m > 0$. We formulate this as a corollary of Theorem II.1.

Corollary II.4. *Let $\varepsilon \leqslant 0$ satisfy $\varepsilon^2 < \Sigma - E_0 - g^2 \sup_x \int |G_x|^2/\omega$, and let ϕ_m be a (normalized) ground state of H_m. Then there exists a constant, $M_\varepsilon < \infty$, depending only on V and Λ_1, uniformly in $m > 0$, such that*

$$\| e^{\varepsilon |x|} \otimes \mathbf{1}_f \phi_m \|^2 \leqslant M_\varepsilon < \infty. \tag{II.22}$$

Proof. Clearly, $E_m \leqslant \langle H_m \rangle_{\psi_{0,1} \otimes \Omega} = E_0$ for $H_{el} \psi_{0,1} = E_0 \psi_{0,1}$, by the variational principle. Thus, the claim follows from Theorem II.1 upon choosing $\Delta := [E_m - 1, E_0]$. ∎

II.4. *Control on Soft Photons*

The main result of this section is the following theorem, which yields control on the number of soft photons and is inspired by [14].

Theorem II.5. *Let ϕ_m be a normalized ground state of H_m, and let N be the photon number operator. Then*

$$\langle \phi_m | \mathbf{1}_{el} \otimes N \phi_m \rangle \leqslant g^2 \int \frac{\| G_x(k) \otimes \mathbf{1}_f \phi_m \|^2 \, dk}{\omega(k)^2}$$

$$\leqslant \| e^{\varepsilon |x|} \otimes \mathbf{1}_f \phi_m \|^2 \cdot \sup_x \left\{ e^{-\varepsilon |x|} \int \frac{|G_x(k)|^2 \, dk}{\omega^2} \right\}, \tag{II.23}$$

where $0 \leqslant \varepsilon \leqslant (\Sigma - E_0 - g^2 \int |G_x|^2/\omega)^{1/2}$.

Proof. By the definition of N,

$$\langle \phi_m | \mathbf{1}_{el} \otimes N \phi_m \rangle = \int \| \mathbf{1}_{el} \otimes a(k) \phi_m \|^2 \, dk. \tag{II.24}$$

Thus the theorem follows from Lemma II.6 below. ∎

LEMMA II.6. *Let ϕ_m be a ground state of H_m. Then*

$$\| \mathbf{1}_{el} \otimes a(k) \phi_m \| \leqslant \omega(k)^{-1} \, \| G_x(k) \otimes \mathbf{1}_f \phi_m \|. \tag{II.25}$$

Proof. We proceed in the spirit of the virial theorem. Commuting an annihilation operator $a(k)$ through H_m, we see that

$$\mathbf{1}_{el} \otimes a(k) \, H_m - (H_m + \omega(k)) \, \mathbf{1}_{el} \otimes a(k) = G_x(k) \otimes \mathbf{1}_f. \tag{II.26}$$

Using that $H_m \phi_m = E_m \phi_m$ and Eq. (II.26), we obtain

$$(H_m - E_m + \omega(k)) \, \mathbf{1}_{el} \otimes a(k) \phi_m = - G_x(k) \otimes \mathbf{1}_f \phi_m.$$

Since $H_m - E_m \geqslant 0$ this, in turn, implies (II.25). ∎

II.5. *Overlap with the Vacuum*

Let P_ϕ stand for the rank-one orthogonal projection on ϕ, i.e., $P_\phi = |\phi\rangle\langle\phi|$. Our results in Section II.4 imply the following lemma.

LEMMA II.7 (Lower Bound on the Overlap). *Let $(E_0 + \Sigma)/2 \leqslant \lambda < \Sigma$ and $P_{el} = \chi_{H_{el} \leqslant \lambda}$. Let ϕ_m be a normalized ground state of H_m, $m > 0$. Suppose that $0 \leqslant \varepsilon \leqslant (\Sigma - E_0 - g^2 \int |G_x|^2/\omega)^{1/2}$ and assume that* (I.47) *holds, with M_ε given in Corollary* II.4. *Then, for all $0 < m$, the ground state ϕ_m of H_m obeys*

$$\langle \phi_m | (P_{el} \otimes P_\Omega) \phi_m \rangle \geqslant 1 - \delta_\varepsilon > 0. \tag{II.27}$$

Proof. Note that (II.27) is equivalent to $\langle \phi_m, (P_{el}^\perp \otimes P_\Omega + \mathbf{1}_{el} \otimes P_\Omega^\perp) \phi_m \rangle \leqslant \delta_\varepsilon$, where we use the notation $P^\perp = \mathbf{1} - P$ for a projection P. Since $P_\Omega^\perp \leqslant N$, it is sufficient to show that

$$\langle \phi_m | (P_{el}^\perp \otimes P_\Omega + \mathbf{1}_{el} \otimes N) \phi_m \rangle \leqslant \delta_\varepsilon, \tag{II.28}$$

in order to establish (II.27). Now we observe that

$$(P_{el}^\perp \otimes P_\Omega) H_m = P_{el}^\perp H_{el} \otimes P_\Omega + (P_{el}^\perp \otimes P_\Omega) \, a(G_x),$$

which implies

$$\begin{aligned}0&=(P_{el}^{\perp}\otimes P_{\Omega})(H_m-E_m)\phi_m\\&=[P_{el}^{\perp}(H_{el}-E_m)\otimes P_{\Omega}]\phi_m+(P_{el}^{\perp}\otimes P_{\Omega})\,a(G_x)\phi_m.\end{aligned}$$

But, $H_{el}P_{el}^{\perp}\geqslant\lambda P_{el}^{\perp}$ and $E_0\geqslant E_m$. Using these estimates and the Schwarz inequality, as in Lemma I.6, we obtain that

$$\begin{aligned}\langle P_{el}^{\perp}\otimes P_{\Omega}\rangle_{\phi_m}&\leqslant(\lambda-E_0)^{-1}\,\langle (H_{el}-E_m)\,P_{el}^{\perp}\otimes P_{\Omega}\rangle_{\phi_m}\\&=-(\lambda-E_0)^{-1}\,\langle (P_{el}^{\perp}\otimes P_{\Omega})\,a(G_x)\rangle_{\phi_m}\\&\leqslant(\lambda-E_0)^{-1}\,\langle P_{el}^{\perp}\otimes P_{\Omega}\rangle_{\phi_m}^{1/2}\,\langle a^{\dagger}(G_x)\,a(G_x)\rangle_{\phi_m}^{1/2}.\end{aligned}$$

Define $\alpha^2:=g^2\sup x\int|G_x|^2$ and $\beta^2:=g^2\int\|G_x(k)\otimes\mathbf{1}_f\phi_m\|^2\,\omega(k)^{-2}\,dk$. Next, we use that

$$\langle a^{\dagger}(G_x)\,a(G_x)\rangle_{\phi_m}\leqslant\alpha^2\langle\mathbf{1}_{el}\otimes N\rangle_{\phi_m}$$

and that, due to Theorem II.5,

$$\langle\mathbf{1}_{el}\otimes N\rangle_{\phi_m}\leqslant\beta^2.$$

The last three inequalities imply

$$\langle P_{el}^{\perp}\otimes P_{\Omega}\rangle_{\phi_m}\leqslant(\lambda-E_0)^{-2}\,\alpha^2\beta^2.$$

The last two inequalities together with Corollary II.4 yield (II.28), and (II.28) implies (II.27). ∎

We remark that, assuming the ground state ψ_0 of H_{el} to be unique, the last two inequalities in the proof of Lemma II.7 imply that

$$\|\phi_m-\phi_0\|^2\leqslant 2(1-|\langle\phi_m,\phi_0\rangle|^2)\leqslant 2\beta^2(1+\alpha^2(\lambda-E_0)^{-2}),$$

where $\phi_0:=\psi_0\otimes\Omega$, upon choosing $\lambda<\min\{E_1,\Sigma\}$.

II.6. *Existence of the Ground State*

In this section we prove the existence of ground states of H. The result below sums up the analysis of the last three chapters.

THEOREM II.8. *Suppose that Hypothesis* 1 *holds with* $0\leqslant\varepsilon\leqslant(\Sigma-E_0-g^2\int|G_x|^2/\omega)^{1/2}$ *and assume* (I.47). *Then, the Hamiltonian* H *has a ground state* ϕ. *As* $g\to 0$, *the ground state energy and the ground state subspace of* H *converge to the unperturbed ground state energy* (= *the ground state energy of the atom*) *and the ground state subspace. If* $\mathcal{H}_{el}=L^2(\mathbb{R}^{3N})$ *and* $G_x(k)=\overline{G_x(-k)}$, *then this ground state is unique. Finally,* $\langle P_{el}\otimes P_{\Omega}\rangle_{\phi}\geqslant 1-\delta_{\varepsilon}>0$, *where* $P_{el} := \chi_{2H_{el} \leqslant \Sigma + E_0}$ *the* (*finite dimensional*) *projection onto the bound states of* H_{el} *with energy below* $\frac{1}{2}\Sigma + \frac{1}{2}E_0$.

Proof. Let H_m and ϕ_m be the same as in the previous three sections. Since the unit ball about 0 in $\mathscr{H}_{el} \otimes \mathscr{F}$ is weakly compact, $\{\phi_m\}$ is weakly compact in $\mathscr{H}_{el} \otimes \mathscr{F}$. Therefore it contains a convergent subsequence which we again denote $\{\phi_m\}$. Let $\phi = \text{w-lim}\, \phi_m$. By (II.27),

$$\langle \phi \,|\, (P_{el} \otimes P_\Omega) \phi \rangle \geqslant 1 - \delta_\varepsilon > 0. \tag{II.29}$$

Thus $\phi \neq 0$.

Let ψ be an arbitrary vector in $\mathscr{D}(H)$. Then

$$\begin{aligned} \langle H\psi \,|\, \phi \rangle &= \lim_{m \to 0} \langle H\psi \,|\, \phi_m \rangle = \lim_{m \to 0} \langle \psi \,|\, H\phi_m \rangle \\ &= \lim_{m \to 0} E_m \langle \psi \,|\, \phi_m \rangle + \lim_{m \to 0} \langle \psi \,|\, (H - H_m) \phi_m \rangle, \end{aligned} \tag{II.30}$$

Using $H - H_m = g(W - W_m)$ in conjunction with (I.59) in Corollary I.7, we find that

$$\begin{aligned} |\langle \psi \,|\, (H - H_m) \phi_m \rangle| &\leqslant g \cdot \sup_x \left\{ \int_{\omega(k) \leqslant m} (1 + \omega(k)^{-1}) \, |G_x(k)|^2 \, dk \right\}^{1/2} \\ &\quad \cdot \| \mathbf{1}_{el} \otimes (H_f + 1)^{1/2} \psi \| \, \|\phi_m\|, \end{aligned} \tag{II.31}$$

which converges to 0, as $m \to 0$, by Lebesgue dominated convergence. Next, we note that

$$E_m \langle \psi \,|\, \phi_m \rangle = \langle \psi \,|\, H_m \phi_m \rangle = \langle H\psi \,|\, \phi_m \rangle + \langle \psi \,|\, (H_m - H) \phi_m \rangle. \tag{II.32}$$

Since, for $\psi \in \mathscr{D}(H)$, $\lim_{m \to 0} \langle H\psi \,|\, \phi_m \rangle = \langle H\psi \,|\, \phi \rangle$, and since $|\langle \psi \,|\, (H_m - H)\, \phi_m \rangle| \to 0$, as $m \to 0$, we conclude that the sequence $\{E_m\}$ has a limit, E, and thus

$$\lim_{m \to 0} E_m \langle \psi \,|\, \phi_m \rangle = E \langle \psi \,|\, \phi \rangle. \tag{II.33}$$

Now, it follows from (II.30) that

$$\langle H\psi \,|\, \phi \rangle = E \langle \psi \,|\, \phi \rangle, \tag{II.34}$$

for arbitrary $\psi \in \mathscr{D}(H)$. Since $\mathscr{D}(H)$ is dense, it follows that $\phi \in \mathscr{D}(H^*) = \mathscr{D}(H)$ and that $H\phi = E\phi$. This proves the existence of the ground state for H. Passing in (II.27) to the limit as $m \to 0$, we conclude that $\langle P_{el} \otimes P_\Omega \rangle_\phi \geqslant 1 - \delta_\varepsilon$.

Finally, as in the proof of Theorem II.2, $H \equiv H_g$ is resolvent norm continuous in g, by Corollary I.7. Hence general perturbation theory implies that the ground state energy $E_0(g)$ of H_g $(=\inf \sigma(H_g))$ is continuous in g and, in particular, converges to E_0, the ground state energy of $H_g=0$. Let P_g denote the projection onto the ground states of H_g. The norm convergence of P_g to $P_{g=0}=\chi_{H_{el}=E_0}\otimes P_\Omega$ (the projection onto groundstates of H_{el} times P_Ω) follows from our estimates in Section II.5 which proves the second statement of Theorem II.8.

Again if $\mathscr{H}_{el}=L^2(\mathbb{R}^{3N})$ then as in the proof of Theorem II.2 (see the end of the proof), the Perron–Frobenius argument implies that the ground state of H is unique, provided $G_x(k)=\overline{G_x(-k)}$. ∎

Theorem II.8 ($\phi :=$ w-lim ϕ_m) together with Theorem II.1 implies

Corollary II.9. *The ground state, ϕ, of H verifies the exponential bound* $\|e^{\alpha|x|}\otimes \mathbf{1}_f\phi\| \leqslant \text{const}\, \delta^{-1/2}\,\|\phi\|$, *provided* $\delta\equiv\Sigma-E_0-g^2\int|G_x|^2/\omega)^{1/2}-\alpha^2>0$.

Note that the exponential bound for the ground state can also be proved directly. Also, we remark that the assumption $\delta_\varepsilon<1$ in Theorem II.8 can be weakened, at the expense of making a more involved argument necessary.

III. POSITIVE COMMUTATORS

In this chapter we begin the study of the continuous spectrum of H_g. To this end, we estimate from below the commutator of H_g with an anti-self-adjoint operator $A=-A^*$. This method, known as the "positive commutator method" originates from [28, 30, 36], with a crucial step in [32] (see [12, 35] for further developments). We refer the reader to [10, 37] for a textbook exposition and remark that the method has been applied to the Spin–Boson model with a positive photon mass, which is related to the model treated here, in [15, 18, 20]. In a forthcoming paper [6], we will present a refined version of the material in this chapter.

Our first choice below for A is $A_1 := \mathbf{1}_{el}\otimes A_f$, where A_f is the second quantization of the dilatation generator on the one-photon space, $L^2(\mathbb{R}^3)$. The estimate of $[A_1, H_g]$ from below allows us to conclude that the spectrum of H_g is absolutely continuous, with no point spectrum, outside a neighbourhood of $\sigma(H_{el})$.

The second choice for A is $A_2 := A_1 + A_{el}\otimes\mathbf{1}_f$, where A_{el} is the dilatation generator on the particle Hilbert space, $L^2(\mathbb{R}^{3N})$. Then we estimate $[A_2, H_g]$ from below in terms of $[A_{el}, H_{el}]$, for small values of g. Our estimate is such that if $[A_{el}, H_{el}]$ is positive, then so is $[A_2, H_g]$. Since, typically, $[A_{el}, H_{el}]$ is positive away from T_{el}, the set of eigenvalues and thresholds of H_{el}, this estimate allows us to conclude that the spectrum of H_g is absolutely continuous, with no point spectrum, outside a small neighbourhood of T_{el}.

In contrast to what we present in Chapter IV, the real axis away from T_{el} covers parts of the half-axis (Σ, ∞). In Chapter IV, we establish absolute continuity of the spectrum of H_g in the interval $[E_0, \Sigma - Cg]$, for some constant, C, assuming that the interaction is analytic with respect to dilatation of the photon momenta.

We come to the assumptions and notation used in this chapter. To begin with, we assume Hypothesis 1. Recall that this means that $H_g = H_0 + gW_x$ with coupling functions $G_x(k)$ in the interaction $W_x = a^\dagger(G_x) + a(G_x)$ that obey

$$\Lambda_1 := \sup_{x \in \mathbb{R}^{3N}} \left\{ \int \left[1 + \frac{1}{\omega(k)} \right] |G_x(k)|^2 \, dk \right\}^{1/2} < \infty. \tag{III.1}$$

Next, we make the definitions of A_1 and A_2 precise. We set

$$A_1 \equiv \mathbf{1}_{el} \otimes A_f := \int a^\dagger(k) \, \tfrac{1}{2}(k \cdot \nabla_k + \nabla_k \cdot k) \, a(k) \, dk, \tag{III.2}$$

which is the second quantization, $d\Gamma(d)$, of the dilatation generator in the one-photon space, $d := \frac{1}{2}(k \cdot \nabla_k + \nabla_k \cdot k)$. Furthermore,

$$A_2 = A_1 + A_{el} \otimes \mathbf{1}_f, \tag{III.3}$$

where A_{el} is the dilatation generator on the atomic Hilbert space $\mathscr{H}_{el}$. Finally, we recall that $E_0, E_1, ..., E_M$ denote the eigenvalues of H_{el} (possibly $M = \infty$), and we set $E_{M+1} := \Sigma$ provided $M < \infty$ and, furthermore, $E_{-1} := -\infty$.

THEOREM III.1. *Fix $j \geqslant -1$ and pick E_s and E_l such that $E_j < E_s < E_l < E_{j+1}$. Assume Hypothesis 1 and, additionally, that*

$$\Theta_{1,\alpha} := \sup_{x \in \mathbb{R}^{3N}} \left\{ e^{\alpha |x|} \int \frac{|k \cdot \nabla_k G_x(k)|^2}{\omega(k)^2} \, dk \right\}^{1/2} < \infty \tag{III.4}$$

holds, where $\alpha \geqslant 0$ is the same as in (II.1). Denote $\Delta := [E_s, E_l]$, and assume that

$$\delta := \operatorname{dist}\{\Delta, \sigma(H_{el})\} = \min\{E_s - E_j, E_{j+1} - E_l\} \geqslant g^{2/5}. \tag{III.5}$$

Then, for $g > 0$ sufficiently small,

$$\chi_\Delta(H_g)[A_1, H_g] \, \chi_\Delta(H_g) \geqslant \delta[1 - O(g^{4/5})] \, \chi_\Delta(H_g)^2. \tag{III.6}$$

Proof. During the proof, we will not display trivial factors in the tensor product $\mathscr{H}_{el} \otimes \mathscr{F}$ and simply write H_f instead of $\mathbf{1}_{el} \otimes H_f$ and H_{el} instead of

$H_{el} \otimes \mathbf{1}_f$, etc. We begin the proof with the remark that the second quantization respects commutators, i.e., for given operators Q, Q' on the one-photon space $L^2(\mathbb{R}^3)$ and given $f \in L^2(\mathbb{R}^3)$, the following relations for their second quantization holds,

$$[d\Gamma(Q), d\Gamma(Q')] = d\Gamma([Q, Q']), \tag{III.7}$$

$$[d\Gamma(Q), a^\dagger(f)] = a^\dagger(Qf), \tag{III.8}$$

as quadratic forms on a suitably chosen domain. Thus, we find by computation,

$$[A_1, H_g] = H_f + g\tilde{W}_x, \tag{III.9}$$

where $\tilde{W}_x = a^\dagger(\tilde{G}_x) + a(\tilde{G}_x)$, with $\tilde{G}_x = k \cdot \nabla_k G_x + \frac{3}{2} G_x$. By (I.60), we have that

$$\pm g W_x \leqslant b H_f + \frac{g^2 \Lambda_1^2}{b}, \tag{III.10}$$

for any $b > 0$. The last two equations show that we have to estimate H_f from below on an appropriate set of vectors. We proceed to do this: Writing $H_g = H_{el} + H_f + g W_x$ and using (III.10), we obtain

$$H_g \leqslant H_{el} + (1+b)\, H_f + \frac{g^2 \Lambda_1^2}{b},$$

which implies that

$$\langle H_f \rangle_\phi \geqslant (1+b)^{-1} \left\langle H_g - H_{el} - \frac{g^2 \Lambda_1^2}{b} \right\rangle_\phi .$$

From now on we assume that $\phi \in \operatorname{Ran} \chi_\Delta(H_g)$. For such vectors we estimate the r.h.s. of the last inequality from below. The key idea of the forthcoming estimate is to use estimates originating in a quantum version of the classical energy conservation law. We have that

$$\langle H_f \rangle_\phi \geqslant (1+b)^{-1} \left\langle E_s - \frac{g^2 \Lambda_1^2}{b} - H_{el} \right\rangle_\phi ,$$

where $E_s = \inf \Delta$. Next, we note that

$$\mathbf{1} = \chi_{H_{el} \leqslant E_j} + \chi_{H_{el} \geqslant E_{j+1}},$$

and, using this decomposition, we obtain the bound

$$\langle H_f \rangle_\phi \geqslant (1+b)^{-1} \left(E_s - E_j - \frac{g^2 \Lambda_1^2}{b} \right) \|\phi\|^2 - (1+b)^{-1} \langle f(H_{el})^2 \rangle_\phi, \tag{III.11}$$

where $f(\mu) = (\mu - E_j)^{1/2} \chi_{\mu \geqslant E_{j+1}}$.

We claim that

$$|\langle f(H_{el})^2 \rangle_\phi| \leqslant \frac{C g^2 \Lambda_1^2}{\delta}. \tag{III.12}$$

To demonstrate (III.12), we point out that it suffices to prove (III.6) for a smooth function, $\tilde{\chi}_\Delta$ instead of χ_Δ, which obeys $\tilde{\chi}_\Delta \equiv 1$ on Δ and such that

$$\operatorname{supp}\{\tilde{\chi}_\Delta\} \subset \left(\frac{E_j + E_s}{2}, \frac{E_l + E_{j+1}}{2} \right). \tag{III.13}$$

We may require without loss of generality that $|\partial^n \tilde{\chi}_\Delta| \leqslant C_n \delta^{-n}$. For notational convenience, we simply assume χ_Δ to have these properties, henceforth. Then, a standard operator calculus estimate using almost analytic functions as in (II.4) (see, e.g., [23]) yields

$$\begin{aligned} \|(H_g + i)[f(H_{el}), \chi_\Delta(H_g)]\| &\leqslant \frac{Cg}{\delta} \|[f(H_{el}), W_x](H_g + i)^{-1}\| \\ &\leqslant \frac{Cg}{\delta^2} (\|f(H_{el}) W_x (H_g + i)^{-1}\| \\ &\quad + \|W_x f(H_{el})(H_g + i)^{-1}\|). \end{aligned} \tag{III.14}$$

Moreover, since both $f(H_{el})$ and H_g are clearly relatively H_0-bounded, the two norms on the right side of (III.14) are bounded by a constant and we obtain

$$\|(H_g + i)[f(H_{el}), \chi_\Delta(H_g)]\| \leqslant \frac{C g \Lambda_1}{\delta}. \tag{III.15}$$

Next, we denote $\hat{E} := \frac{1}{2}(E_l + E_{j+1})$ and observe that (III.13) implies that

$$\chi_\Delta(H_g) \, \chi_{H_g \geqslant \hat{E}} = 0. \tag{III.16}$$

These two relations, (III.15) and (III.16), show that

$$\langle f(H_{el}) \, \chi_{H_g \geqslant \hat{E}} \, f(H_{el}) \rangle_\phi = O(g \delta^{-1} \Lambda_1), \tag{III.17}$$

$$\langle f(H_{el}) \, \chi_{H_g \geqslant \hat{E}} H_g \, f(H_{el}) \rangle_\phi = O(g \delta^{-1} \Lambda_1), \tag{III.18}$$

which, in turn, implies that

$$\langle f(H_{el})\, \chi_{H_g \leqslant \hat{E}}\, f(H_{el}) \rangle_\phi = \langle f(H_{el})^2 \rangle_\phi + O(g\delta^{-1}\Lambda_1), \tag{III.19}$$

$$\langle f(H_{el})\, \chi_{H_g \leqslant \hat{E}} H_g\, f(H_{el}) \rangle_\phi = \langle f(H_{el})\, H_g\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1), \tag{III.20}$$

Using these two equations, we estimate

$$\begin{aligned} \hat{E}\langle f(H_{el})^2 \rangle_\phi &= \hat{E}\langle f(H_{el})\, \chi_{H_g \leqslant \hat{E}}\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1) \\ &\geqslant \langle f(H_{el})\, \chi_{H_g \leqslant \hat{E}} H_g\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1) \\ &= \langle f(H_{el})\, H_g\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1). \end{aligned}$$

Due to (III.10) with $b=1$, we have that $H_f + gW_x \geqslant -g^2\Lambda_1^2$. Using this fact, (III.19), (III.20), and that $f(\mu) \neq 0$ only for $\mu \geqslant E_{j+1}$, we derive the bound

$$\begin{aligned} \hat{E}\langle f(H_{el})^2 \rangle_\phi &= \hat{E}\langle f(H_{el})\, \chi_{H_g \leqslant \hat{E}}\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1) \\ &\geqslant \langle f(H_{el})\, H_g \chi_{H_g \leqslant \hat{E}}\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1) \\ &= \langle f(H_{el})\, H_g\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1) \\ &\geqslant \langle f(H_{el})(H_{el} - g^2\Lambda_1)\, f(H_{el}) \rangle_\phi + O(g\delta^{-1}\Lambda_1) \\ &\geqslant (E_{j+1} - g^2\Lambda_1)\langle f(H_{el})^2 \rangle_\phi + O(g\delta^{-1}\Lambda_1). \end{aligned} \tag{III.21}$$

Choosing g sufficiently small such that $4g^2\Lambda_1^2 \leqslant Cg \leqslant \delta$, we obtain $E_{j+1} - \hat{E} - g^2\Lambda_1 > \delta/4$, which implies (III.12). Together with (III.11), (III.12) yields that

$$\langle H_f \rangle_\phi \geqslant \left(\frac{1}{1+b} \right) \left(\delta - \frac{g^2\Lambda_1^2}{b} - \frac{Cg^2\Lambda_1^2}{\delta^2} \right) \|\phi\|^2. \tag{III.22}$$

Recalling the definition of $\widetilde{W}_x$ (see the line after Eq. (III.9)) and using (I.60), we obtain

$$|\langle g\widetilde{W}_x \rangle_\phi| \leqslant \tilde{b}\langle H_f \rangle_\phi + \frac{g^2}{\tilde{b}} (\Theta_{1,\alpha})^2 \|\phi\|^2, \tag{III.23}$$

for any $\tilde{b} > 0$. Adding up (III.22) and (III.23) according to (III.9), we arrive at

$$\begin{aligned} \frac{\langle H_f + g\widetilde{W}_x \rangle_\phi}{\|\phi\|^2} &\geqslant \left(\frac{1-\tilde{b}}{1+b} \right) \left(\delta - \frac{g^2\Lambda_1^2}{b} - \frac{Cg^2\Lambda_1^2}{\delta^2} \right) - \frac{g^2}{\tilde{b}} (\Theta_{1,\alpha})^2 \\ &= \delta \left[1 + O\left(\frac{g^2}{b\delta} + \frac{g^2}{\tilde{b}\delta} + \frac{g^2}{\delta^3} + b \right) \right]. \end{aligned} \tag{III.24}$$

The claim (III.6) follows now from choosing $b := \tilde{b} := g^{4/5}$, since we assumed in (III.5) that $\delta \geqslant g^{2/5}$. ∎

Note that estimate (III.6) is more elementary than what is known as Mourre estimate; (the energy intervals are not shrunk in order to handle the contribution of the interaction) It is closer to estimates derived in [28, 30].

In order to derive the desired results from Theorem III.1, we use Mourre theory. We verify the applicability of this theory: Eqs. (III.9) and (III.23) imply that $[A_1, H_g]$ is H_g-bounded. Next, we iterate (III.9) and obtain the second commutator,

$$[A_1, [A_1, H_g]] = H_f + a^\dagger((k \cdot \nabla_k + 3/2)^2 G_x) + a((k \cdot \nabla_k + 3/2)^2 G_x). \tag{III.25}$$

An analogue of (III.23) now implies the H_g-boundedness of this second commutator. Note that only for this estimate, we need

$$\sup_x |(k \cdot \nabla_k)^2 G_x(k)| \, \omega(k)^{-1/2} \in L^2(\mathbb{R}^3).$$

Also, the relative boundedness of $[A_1, H_g]$ only requires $\sup_x |k \cdot \nabla_k G_x(k)| \, \omega(k)^{-1/2} \in L^2(\mathbb{R}^3)$, and for Theorem III.1, it would actually suffice to assume $\sup_x \{ e^{-\alpha |x|} |k \cdot \nabla_k G_x(k)| \, \omega(k)^{-1/2} \} \in L^2(\mathbb{R}^3)$, for some $\alpha > 0$ sufficiently small.

The relative H_g-boundedness of $[A_1, H_g]$ and $[A_1, [A_1, H_g]]$ and Theorem III.1 yield (see e.g. [23]).

Lemma III.2. *Assume that Hypothesis* 1 *and Condition* (III.4) *hold and, additionally, that, for* $n = 1, 2$,

$$\Theta_n := \sup_{x \in \mathbb{R}^{3N}} \left\{ \int \frac{|(k \cdot \nabla_k)^n G_x(k)|^2}{\omega(k)^2} \, dk \right\}^{1/2} < \infty. \tag{III.26}$$

Then, for sufficiently small $g > 0$, *the spectrum of* H_g *in* $\Omega := \{ \lambda \in \mathbb{R} \mid \operatorname{dist}\{\lambda, \sigma(H_{el})\} > g^{2/5} \}$ *is absolutely continuous. Moreover, for any* $\Delta \subset \Omega$, *the limiting absorption principle holds*: $(H - z)^{-1} \chi_\Delta(H_g)$, *as a map from* $\langle A_1 \rangle^{-\mu} \mathscr{H}$ *to* $\langle A_1 \rangle^{\mu} \mathscr{H}$, $\mu > \frac{1}{2}$, *is bounded and norm continuous, as* z *approaches the real axis.*

Next, we consider $\sigma_{\text{cont}}(H_{el})$. Recall that A_{el} is the dilatation generator in the atomic Hilbert space $\mathscr{H}_{el}$. We require Hypothesis 1 and assume that the Mourre estimate holds for H_{el} and A_{el}. Specifically, we assume that, given $\mu \in \sigma_{\text{cont}}(H_{el}) \backslash T_{el}$, *and given* $\varepsilon > 0$, *there is some* $\delta > 0$ *such that*

$$\bar{\chi}_\Delta(H_{el}) [A_{el}, H_{el}] \, \bar{\chi}_\Delta(H_{el}) \geqslant (\theta_{el} - \varepsilon) \, \bar{\chi}_\Delta(H_{el})^2, \tag{III.27}$$

provided $\Delta = \mu + [-\delta, \delta]$. Here,

$$\theta_{el} := \operatorname{dist}\{\Delta, T_\Delta\}, \qquad T_\Delta := T_{el} \cap (-\infty, \sup \Delta), \tag{III.28}$$

where T_{el} are the thresholds and eigenvalues of H_{el}, and

$$\bar{\chi}_\Delta(H_{el}) := \chi_{\Delta \cap [\Sigma, \infty)}(H_{el}) = \chi_\Delta(H_{el}) P_{el}^{\mathrm{cont}}.$$

Note that, by passing from δ to $\delta/2$, we may assume that

$$\bar{\chi}_{\Delta - \nu}(H_{el})[A_{el}, H_{el}]\, \bar{\chi}_{\Delta - \nu}(H_{el}) \geqslant (\theta_{el} - \varepsilon)\, \bar{\chi}_{\Delta - \nu}(H_{el})^2, \tag{III.29}$$

for any $-\delta \leqslant 2\nu \leqslant \delta$.

THEOREM III.3. *Assume that Hypothesis* 1 *holds and, additionally, that*

$$\hat{\Theta}_1 := \sup_{x \in \mathbb{R}^{3N}} \left\{ \int \frac{|(k \cdot \nabla_k - x \cdot \nabla_x)\, G_x(k)|^2}{\omega(k)^2}\, dk \right\}^{1/2} < \infty. \tag{III.30}$$

Let μ, ε, Δ, *and* θ_{el} *be as specified in* (III.29). *Set* $\Delta' := \mu + \frac{1}{2}[-\delta, \delta]$. *Then*

$$\chi_{\Delta'}(H_g)[A_2, H_g]\, \chi_{\Delta'}(H_g) \geqslant \theta' \chi_{\Delta'}(H_g)^2, \tag{III.31}$$

where

$$\theta' = \min\left\{\theta_{el}, \frac{\delta}{4}\right\} - \varepsilon - Cg^2 - \frac{Cg}{\delta}. \tag{III.32}$$

Proof. Using (III.7) and (III.8), we find that

$$[A_2, H_g] = [A_{el}, H_{el}] + H_f + g\hat{W}_x, \tag{III.33}$$

where

$$\hat{W}_x = \tilde{W}_x + [A_{el}, W_x] = a^\dagger(F_x) + a(F_x) \tag{III.34}$$

with

$$F_x = k \cdot \nabla_k G_x - x \cdot \nabla_x G_x + \tfrac{3}{2} G_x. \tag{III.35}$$

Using (III.10), we obtain

$$[A_2, H_g] \geqslant [A_{el}, H_{el}] + \tfrac{1}{2} H_f - 2g^2(\Lambda_1 + \hat{\Theta}_1)^2.$$

By multiplying this inequality from the left and the right by $\chi_\Delta(H_0)$, where $H_0 = H_{el} + H_f$, and using the direct integral representation of the r.h.s., we obtain

$$\chi_\Delta(H_0)[A_2, H_g]\,\chi_\Delta(H_0) \geqslant \int_{v \geqslant 0}^{\oplus} \chi_{\Delta - v}(H_{el}) \left([A_{el}, H_{el}] + \frac{v}{2} \right) \chi_{\Delta - v}(H_{el})\, dv$$
$$-2g^2(\Lambda_1 + \hat{\Theta}_1)^2\, \chi_\Delta(H_0)^2. \tag{III.36}$$

Now, we break up the integral on the r.h.s. of (III.36) into one over $[0, \delta]$ and one over $[\delta, \infty]$. Due to (III.29), the first integral is bounded from below by

$$\int_{0 \leqslant v \leqslant \delta}^{\oplus} \chi_{\Delta - v}(H_{el}) \left([A_{el}, H_{el}] + \frac{v}{2} \right) \chi_{\Delta - v}(H_{el})\, dv$$
$$\geqslant (\theta_{el} - \varepsilon) \int_{0 \leqslant v \leqslant \delta}^{\oplus} \chi^2_{\Delta - v}(H_{el})\, dv, \tag{III.37}$$

while the second one is bounded by

$$\int_{v \geqslant \delta}^{\oplus} \chi_{\Delta - v}(H_{el}\Big(([A_{el}, H_{el}] + \frac{v}{2} \Big) \chi_{\Delta - v}(H_{el})\, dv$$
$$\geqslant \left(\frac{\delta}{4} - \varepsilon \right) \int_{0 \leqslant v \leqslant \delta}^{\oplus} \chi^2_{\Delta - v}(H_{el})\, d. \tag{III.38}$$

Thus, we obtain

$$\chi_\Delta(H_0)[A_2, H_g]\,\chi_\Delta(H_0)$$
$$\geqslant \left(\min\left\{ \theta_{el}, \frac{\delta}{4} \right\} - \varepsilon - 2g^2(\Lambda_1 + \hat{\Theta}_1)^2 \right) \chi_\Delta(H_0)^2. \tag{III.39}$$

Let $\Delta' := \mu + \frac{1}{2}[-\delta, \delta]$ such that $\operatorname{dist}\{\Delta', \mathbb{R} \backslash \Delta\} = \delta/2$. Using operator calculus (see, e.g., [23]), we find that

$$\|\chi_{\Delta'}(H_g)(1 - \chi_\Delta(H_0))\| \leqslant \frac{Cg\Lambda_1}{\delta}.$$

This, together with (III.39), yields

$$
\begin{aligned}
&\chi_{\Delta'}(H_g)[A_2, H_g]\,\chi_{\Delta'}(H_g)\\
&\quad \geqslant \chi_{\Delta'}(H_g)\,\chi_{\Delta}(H_0)[A_2, H_g]\,\chi_{\Delta}(H_0)\,\chi_{\Delta'}(H_g) - \frac{Cg\Lambda_1}{\delta}\,\chi^2_{\Delta'}(H_g)\\
&\quad \geqslant \left(\min\left\{\theta_{el}, \frac{\delta}{4}\right\} - \varepsilon - 2g^2(\Lambda_1 + \hat{\Theta}_1)^2\right)\chi_{\Delta'}(H_g)\,\chi_{\Delta}(H_0)^2\,\chi_{\Delta'}(H_g)\\
&\quad\quad - \frac{Cg\Lambda_1}{\delta}\,\chi^2_{\Delta'}(H_g)\\
&\quad \geqslant \left(\min\left\{\theta_{el}, \frac{\delta}{4}\right\} - \varepsilon - 2g^2(\Lambda_1 + \hat{\Theta}_1)^2 - \frac{Cg\Lambda_1}{\delta}\right)\chi^2_{\Delta'}(H_g). \qquad \text{(III.40)}
\end{aligned}
$$

This inequality implies (III.31) upon replacing δ by $\delta/2$. ∎

As in the discussion of Lemma III.2 above, A_2 maps a core of H_g into itself, and $[A_2, H_g]$ and $[A_2, [A_2, H_g]]$ are H_g-bounded, provided $\hat{\Theta}_n$ (defined below) is finite, for $n = 1, 2$. Thus, Theorem III.3 implies (see [23])

THEOREM III.4. *Assume that Hypothesis* 1 *holds and, additionally, that, for* $n = 1, 2$,

$$
\hat{\Theta}_n := \sup_{x \in \mathbb{R}^{3N}} \left\{ \int \frac{|(k \cdot \nabla_k - x \cdot \nabla_x)^n\, G_x(k)|^2}{\omega(k)^2}\, dk \right\}^{1/2} < \infty. \qquad \text{(III.41)}
$$

Pick $\mu \in [\sigma, \infty)$ *away from the thresholds of* H_{el} *and* $\varepsilon > 0$. *Let* δ, Δ, *and* θ_{el} *be as specified in* (III.29). *Set* $\Delta' := \mu + \frac{1}{2}[-\delta, \delta]$. *Then, for* $g \cdot \max\{1, \delta^{-2}\} > 0$ *sufficiently small, the spectrum of* H_g *in* Δ' *is absolutely continuous. Moreover, the limiting absorption principle holds*: $(H_g - z)^{-1}\chi_{\Delta'}(H_g)$, *as a map from* $\langle A_2\rangle^{-\mu}\mathscr{H}$ *to* $\langle A_2\rangle^{\mu}\mathscr{H}$, $\mu > \frac{1}{2}$, *is bounded and norm continuous as* z *approaches the real axis.*

Additionally assuming that $\delta \geqslant c\theta_{el}$ for some small constant, $c > 0$, independent of μ and ε, we obtain the following corollary

COROLLARY III.5. *Assume that Hypothesis* 1 *and* (III.41) *hold. Pick* $\varepsilon > 0$ *and suppose there exists a small contant*, $c > 0$, *such that* (III.27)–(III.28) *holds for any interval*, Δ, *with* $|\Delta| \geqslant c\theta_{el}$. *Then there exists a constant*, C, *such that the spectrum of* H_g *in* $\mathscr{A}_g := \mathbb{R}\backslash(T_{el} + [-Cg, Cg])$ *is absolutely continuous.*

For more precise results along the same lines see [6].

Theorem III.4 above completes our study of the spectrum of H_g outside a small neighbourhood of the eigenvalues and thresholds of H_{el}. We proceed now to investigating the nature of the spectrum of H_g in

$(E_0, \Sigma - O(g))$. This part is harder and makes use of some elements of the renormalization group construction. In the next chapter, we begin with developing some basic technical tools.

IV. THE FESHBACH MAP AND INSTABILITY OF EXITED STATES

In this section we define (in an abstract context) the Feshbach map and establish a key "isospectral property". This map will provide a tool for studying the spectrum of the Hamiltonian H near the atomic energies E_j.

IV.1. *The Feshbach Map*

Let $\mathscr{H}$ be a separable Hilbert space, and let P be a bounded, but not necessarily orthogonal projection on $\mathscr{H}$ (i.e., $P^2 = P$, but possibly $P^* \neq P$). We set $\bar{P} = \mathbf{1} - P$. For any densely defined, closed operator H on $\mathscr{H}$, whose domain contains the range of P, we define

$$H_P := PHP \qquad \text{and} \qquad H_{\bar{P}} := \bar{P}H\bar{P}. \tag{IV.1}$$

For an operator A, let $\rho(A)$ denote its resolvent set, i.e., the set of complex numbers z such that $A - z\mathbf{1}$ has a bounded inverse. We view $H_{\bar{P}}$ as an operator on $\bar{P}\mathscr{H}$ and assume that $0 \in \rho(H_{\bar{P}})$, i.e.,

$$(H_{\bar{P}})^{-1} \qquad \text{exists on } \bar{P}\mathscr{H} \text{ and is bounded,} \tag{IV.2}$$

and, furthermore,

$$\|\bar{P}(H_{\bar{P}})^{-1}\,\bar{P}HP\| < \infty, \qquad \text{and} \qquad \|PH\bar{P}(H_{\bar{P}})^{-1}\,\bar{P}\| < \infty. \tag{IV.3}$$

We define the *Feshbach map*, $f_P(H)$, by

$$f_P(H) = (PHP - PH\bar{P}(H_{\bar{P}})^{-1}\,\bar{P}HP)|_{\operatorname{Ran} P}, \tag{IV.4}$$

provided $0 \in \rho(H_{\bar{P}})$.

Next we define

$$S_P = P - \bar{P}(H_{\bar{P}})^{-1}\,\bar{P}HP. \tag{IV.5}$$

Then

$$\operatorname{Ker}\{S_P\} = \operatorname{Ker}\{P\} \tag{IV.6}$$

Indeed, we have the identity

$$P\varphi = [\mathbf{1} + \bar{P}(H_{\bar{P}})^{-1}\,\bar{P}HP]\,S_P\varphi. \tag{IV.7}$$

Hence, $S_P\varphi = 0$ if and only if $P\varphi = 0$. Shifting $H \mapsto H - z$ by a spectral parameter, $z \in \mathbb{C}$, we have

THEOREM IV.1. *Under Assumption* (IV.3), *and for* $z \in \rho(H_{\bar{P}})$, *we have that*

$$z \in \sigma_{\#}(H) \Leftrightarrow 0 \in \sigma_{\#}(H - z) \Leftrightarrow 0 \in \sigma_{\#}[f_P(H - z)], \tag{IV.8}$$

where $\sigma_{\#} = \sigma$ *or* $\sigma_{\#} = \sigma_{\mathrm{pp}}$. *Moreover, eigenfunctions of* $H - z$ *and* $f_P(H - z)$ *are related by*

$$\mathrm{Ker}\{(H - z)\, S_P(z)\} = \mathrm{Ker}\{f_P(H - z)\}, \tag{IV.9}$$

where $S_P(z) := P - \bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P}HP$, *and*

$$P\,\mathrm{Ker}(H - z) = \mathrm{Ker}\{f_P(H - z)\}. \tag{IV.10}$$

These relations imply, in particular, that

$$\dim\mathrm{Ker}\{H - z\} = \dim\mathrm{Ker}\{f_P(H - z)\}. \tag{IV.11}$$

Proof. Statements (IV.8) and (IV.9) follow from the following identities:

$$(H - z)\, S_P(z) = f_P(H - z)\, P, \tag{IV.12}$$

$$P(H - z)^{-1}\, P = [f_P(H - z)]^{-1}, \tag{IV.13}$$

on $\mathrm{Ran}\{P\}$, and

$$\begin{aligned}(H - z)^{-1} = {} & [f_P(H - z)]^{-1}\, P - [f_P(H - z)]^{-1}\, PH\bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P} \\ & - \bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P}HP(f_P(H) - z)^{-1} + \bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P} \\ & + \bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P}HP(f_P(H) - z)^{-1}\, PH\bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P}.\end{aligned} \tag{IV.14}$$

These identities are proved by elementary algebraic manipulations based on the second resolvent identify and on the representation

$$H = H_P + H_{\bar{P}} + PH\bar{P} + \bar{P}HP.$$

Identity (IV.12) implies (IV.9) and identities (IV.13) and (IV.14) imply (IV.8). In the latter case, the argument proceeds as follows: Let $z \in \rho(H) \cap \rho(H_{\bar{P}})$. Then one shows that the l.h.s. of (IV.13) defines the inverse of $f_P(H - z)$ and therefore $0 \in \rho[f_P(H - z)]$. Next, suppose that $0 \in \rho[f_P(H - z)]$ and $z \in \rho(H_{\bar{P}})$. Then the r.h.s. of (IV.14) defines the inverse of $H - z$, so $z \in \rho(H)$. It remains to prove (IV.10). Let $z \in \sigma_{\mathrm{pp}}(H)$ and $0 \neq \psi \in \mathrm{Ker}\{H - z\}$. Projecting the equation $(H - z)\,\psi = 0$ on Ran P and on Ran $\bar{P}$, we obtain

$$(H_P - z)\, P\psi - PH\bar{P}\psi = 0, \qquad (H_{\bar{P}} - z)\, \bar{P}\psi = -\bar{P}HP\psi.$$

Solving the second equation for $\bar{P}\psi$, we obtain that

$$\bar{P}\psi = -\bar{P}(H_{\bar{P}} - z)^{-1}\, \bar{P}HP\psi.$$

Substituting this identity into the first equation, we find that $f_P(H - z)\, P\psi = 0$ and hence

$$P\,\mathrm{Ker}\{H - z\} \subseteq \mathrm{Ker}\{f_P(H - z)\}.$$

Conversely, if $0 \neq \varphi = P\varphi \in \mathrm{Ker}\{f_P(H - z)\}$ then $S_P(z)\,\varphi \in \mathrm{Ker}\{H - z\}$, by (IV.12). Thus $P\varphi = PS_P(z)\,\varphi \in P\,\mathrm{Ker}\{H - z\}$. ∎

Theorem IV.1 establishes the basic properties of the Feshbach map crucial for the methods we develop in subsequent sections. We summarize Theorem IV.1 by saying that the map f_P is isospectral.

Let us mention a further important property of the Feshbach map which follows directly from (IV.13):

$$f_{P_1} \circ f_{P_2} = f_{P_1 P_2}, \tag{IV.15}$$

provided $[P_1, P_2] = 0$. In Chapter V, we use Identity (IV.15) with $P_1 P_2 = P_2 P_1 = P_2$, in which case it may be interpreted as a semigroup property.

IV.2. *Instability of Excited States from Fermi's Golden Rule*

Our first application of the Feshbach map to the operator H_g from (I.22) is to eliminate the particle degrees of freedom. Simultaneously, we shall eliminate the degree of freedom of the photon field corresponding to field energies $\geqslant \rho_0$, for some $\rho_0 > 0$ to be specified below. This reduces our spectral problem on $\mathscr{H}_{el} \otimes \mathscr{F}$ to one on the Hilbert space $\mathbb{C}^{N_j} \otimes \{\chi_{H_f < \rho_0} \mathscr{F}\}$, where $\chi_{H_f < \rho_0} \mathscr{F} = \mathrm{Ran}\{\chi_{H_f < \rho_0}\}$ and $\mathbb{C}^{N_j} = \mathrm{span}\{\psi_{j,\ell} \mid 1 \leqslant \ell \leqslant N_j\}$ is the subspace in H_{el} spanned by the eigenfunctions $\psi_{j,\ell}$ of H_{el} corresponding to the energy E_j, i.e., $H_{el}\psi_{j,\ell} = E_j \psi_{j,\ell}$.

We recall from (I.22) that H_g is given by

$$H_g = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + W_g, \tag{IV.16}$$

where $W_g = gW_1 + g^2 W_2$, and

$$W_1 = \int \{G_{1,0}(k) \otimes a^\dagger(k) + G_{0,1}(k) \otimes a(k)\}\, dk, \tag{IV.17}$$

$$W_2 = \int \{G_{2,0}(k,k') \otimes a^\dagger(k)\, a^\dagger(k') + G_{0,2}(k,k') \otimes a(k)\, a(k') + G_{1,1}(k,k') \otimes a^\dagger(k)\, a(k')\}\, dk. \tag{IV.18}$$

Here, $G_{m,n}$ are operator-valued functions whose specification we recall below. Our purpose is to prove the following theorem.

THEOREM IV.2. (a) *For $g > 0$ sufficiently small, the operator H_g has absolutely continuous spectrum in each interval $[\frac{2}{3}E_{j-1} + \frac{1}{3}E_j, \frac{1}{3}E_j + \frac{2}{3}E_{j+1}]$, where $j \geqslant 1$ is such that the $N_j \times N_j$ selfadjoint matrix*

$$(A_j)_{\mu\nu} := \int_{-\infty}^{E_j - 0} \int dk \langle G_{10}(k)\, \psi_{j,\mu} \,|\, d\chi_{H_{el} \leqslant E}\, G_{10}(k)\, \psi_{j,\nu}\rangle\, \delta[\omega(k) - E_j + E] \tag{IV.19}$$

is strictly positive definite;

(b) *for vectors u, v analytic for $\mathbf{1}_{el} \otimes iA_f$, the matrix elements $\langle u | (H_g - z)^{-1} v\rangle$ of the resolvent $(H_g - z)^{-1}$ have an analytic continuation from $\mathbb{C}_+$ into*

$$\mathscr{I}_j := [\tfrac{2}{3}E_{j-1} + \tfrac{1}{3}E_j, \tfrac{1}{3}E_j + \tfrac{2}{3}E_{j+1}] + i(\mathbb{R}_+ - \gamma_j), \tag{IV.20}$$

where $\gamma_j := g^2 a_j$, with a_j being the smallest eigenvalue of A_j.

Clearly, Theorem IV.2(a) is a consequence of Theorem IV.2(b) which, in turn, is a consequence of Theorem IV.3 below. In the proof, we use the method of complex dilatations described in Section I.5. Then we apply the Feshbach map to the operator $H_g(\theta)$, in order to eliminate the particle degrees of freedom. Simultaneously, we shall eliminate the degrees of freedom of the photon field corresponding to field energies $\geqslant \rho_0$, for some $\rho_0 > 0$ to be specified below. This reduces our spectral problem to one on the Hilbert space $\mathbb{C}^{N_j} \otimes \{\chi_{H_f < \rho_0} \mathscr{F}\}$, where $\chi_{H_f < \rho_0} \mathscr{F} = \mathrm{Ran}\{\chi_{H_f < \rho_0}\}$ and $\mathbb{C}^{N_j} = \mathrm{span}\{\psi_{j,\ell} \,|\, 1 \leqslant \ell \leqslant N_j\}$ is the subspace in $\mathscr{H}_{el}$ spanned by the eigenfunctions $\psi_{j,\ell}$ of H_{el} corresponding to the energy E_j, i.e., $H_{el}\psi_{j,\ell} = E_j \psi_{j,\ell}$.

IV.2.1. *Complex dilations.* We recall from (I.65) that a complex dilatation $U_f(\theta)$ transforms H_g into $H_g(\theta) \equiv U_f(\theta)\, H_g\, U_f(\theta)^{-1}$, given by

$$H_g(\theta) = H_{el} \otimes \mathbf{1} + \mathrm{e}^{-\theta}(\mathbf{1} \otimes H_f) + W_g(\theta), \tag{IV.21}$$

where $W_g(\theta) = gW_1(\theta) + g^2 W_2(\theta)$ results from (IV.17), (IV.18) by the replacement $G_{m,n}(\underline{k}) \to G^{(\theta)}_{m,n}(\underline{k}) := \mathrm{e}^{-(3\theta/2)(m+n)} G_{m,n}(\mathrm{e}^{-\theta}\underline{k})$ with $1 \leqslant m+n \leqslant 2$ and $\underline{k} \in \mathbb{R}^{3(m+n)}$.

Throughout this chapter we require Hypothesis 2, i.e., for $m+n=1$ and $k \in \mathbb{R}^3$, we assume $G^{(\theta)}_{m,n}(k)$ to be analytic functions with values in the quadratic forms defined on $D[(-\Delta_x)^{1/4}]$, and for $m+n=2$ and $\underline{k} \in \mathbb{R}^3 \times \mathbb{R}^3$, we assume $G^{(\theta)}_{m,n}(\underline{k})$ to be analytic functions with values in the bounded operators on $\mathscr{H}_{el}$, obeying (I.35)–(I.39), for some $\beta > 0$.

THEOREM IV.3. *We require Hypothesis 2 and assume that $E_j > E_0$ is an isolated eigenvalue of H_{el} of degeneracy $N_j < \infty$ with eigenvectors $\psi_{j,1}, ..., \psi_{j,N_j}$. Assume that A_j in* (IV.19) *is positive definite, i.e., $A_j \geqslant a_j \mathbf{1} > 0$, and define $\gamma_j := g^2 a_j$. Then, for $g > 0$ sufficiently small and some constant C, the set*

$$\mathscr{I}_j := [\tfrac{2}{3}E_{j-1} + \tfrac{1}{3}E_j, \tfrac{1}{3}E_j + \tfrac{2}{3}E_{j+1}] - i(\gamma_j - Cg^{2+\beta/(2+\beta)}) + i\mathbb{R}_+, \qquad \text{(IV.22)}$$

is contained in the resolvent set $\rho[H_g(\theta)]$ of $H_g(\theta)$.

Before we proceed to the proof of Theorem IV.3 (which we give in Subsection IV.4.1), we introduce some more notation and derive some preliminary lemmata which pave the way to Theorem IV.3.

IV.3. *Elimination of Particle Degrees of Freedom*

In this section, we verify the hypotheses of Theorem IV.1 which allow us to apply the isospectral Feshbach map (see Corollary IV.5), after introducing some more notation and proving a preliminary technical estimate.

To begin with, we fix $\theta =: i\vartheta$ to be some small, purely imaginary number (in spite of the fact that the connection between Theorem IV.2(b) above and Theorem IV.3 is given by the analytic continuation of matrix elements of the resolvent as a function of θ). Thus, we shall not display the θ-dependence of $G^{(\theta)}_{m,n}$ anymore and simply write $G_{m,n}$, instead. Moreover, our estimates below do not reflect the dependence on $\vartheta > 0$, either, although they are not uniform as $\vartheta \to 0$.

Let

$$\delta = \min\{|E_j - E_{j-1}|, |E_{j+1} - E_j|\} > 0, \qquad \text{(IV.23)}$$

where possibly $E_{j+1} = \Sigma$. Since V is bounded relative to $-\Delta$, both operators

$$(|H_{el} - E_j| + \mathbf{1})(-\Delta + \mathbf{1})^{-1} \qquad \text{and} \qquad (-\Delta + \mathbf{1})^{-1}(|H_{el} - E_j| + \mathbf{1}),$$

are bounded on $\mathscr{H}_{el}$. Thus, by Hypothesis 2, there exists a function J such that, for all $k, k' \in \mathbb{R}^3$ and $m+n=1$,

$$\|(|H_{el}-E_j|+\mathbf{1})^{-1/4}\, G_{m,n}(k)(|H_{el}-E_j|+\mathbf{1})^{-1/4}\| \leqslant J(k), \quad \text{(IV.24)}$$

and, for $m+n=2$,

$$\|G_{m,n}(k,k')\| \leqslant J(k)\, J(k'). \quad \text{(IV.25)}$$

Next, we choose some $0<\tau<1$ and fix

$$\rho_0 := g^{2-2\tau}, \quad \text{(IV.26)}$$

such that, for $g>0$ sufficiently small,

$$\rho_0 \leqslant \frac{\delta}{2} \sqrt{1-\cos \vartheta}. \quad \text{(IV.27)}$$

We define

$$P_0 := P_{el,j} \otimes \chi_{H_f<\rho_0} \equiv \sum_{\ell=1}^{N_j} |\psi_{j,\ell}\rangle\langle\psi_{j,\ell}| \otimes \chi_{H_f<\rho_0}. \quad \text{(IV.28)}$$

Then

$$\bar{P}_0 := \mathbf{1} - P_0 = \bar{P}_0^{(1)} + \bar{P}_0^{(2)}, \quad \text{(IV.29)}$$

where

$$\bar{P}_0^{(1)} = P_{el,j} \otimes \chi_{H_f \geqslant \rho_0}, \quad \text{(IV.30)}$$

$$\bar{P}_0^{(2)} = \bar{P}_{el,j} \otimes \mathbf{1}_f. \quad \text{(IV.31)}$$

In order to control the spectrum of $H_g(\theta)$ we appeal to Theorem IV.1:

$$z \in \sigma[H_g(\theta)] \cap D(E_j, \rho_0/2) \Leftrightarrow 0 \in \sigma[f_{P_0}(H_g(\theta)-z)] \cap D(E_j - z, \rho_0/2), \quad \text{(IV.32)}$$

where f_{P_0} is the Feshbach map defined in (IV.4). The equivalence (IV.32) is valid provided z belongs to the resolvent set of $\bar{P}_0 H_g(\theta) \bar{P}_0$ (see (IV.2)) and Inequalities (IV.3) hold. To verify this hypotheses, we require the following lemma.

LEMMA IV.4. *Suppose that $z \in D(E_j, \rho_0/2)$, i.e., that $|E_j - z| \leqslant \rho_0/2$, and assume that $\Lambda_{1+\beta}, \delta < \infty$, and $0<\vartheta<\pi/2$. Then, for some constant $C_0 \equiv C_0(\Lambda_{1+\beta}, \delta, \vartheta) < \infty$ and $l=1,2$,*

$$\| |H_0 - z|^{-1/2} \, \bar{P}_0 W_l \bar{P}_0 \, |H_0 - z|^{-1/2} \| \leqslant C_0 \cdot \rho_0^{-1/2}, \tag{IV.33}$$

$$\| |H_0 - z|^{-1/2} \, \bar{P}_0 W_l P_0 \| \leqslant C_0, \tag{IV.34}$$

$$\| P_0 W_l \bar{P}_0 \, |H_0 - z|^{-1/2} \| \leqslant C_0. \tag{IV.35}$$

Proof. First, we introduce

$$K := |H_{el} - E_j| \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + \rho_0 \tag{IV.36}$$

and observe that

$$\left\| \bar{P}_0 \frac{K}{H_0 - z} \right\| = \max \left\{ \left\| \bar{P}_0^{(1)} \frac{K}{H_0 - z} \right\|, \left\| \bar{P}_0^{(2)} \frac{K}{H_0 - z} \right\| \right\},$$

and

$$\left\| \bar{P}_0^{(1)} \frac{K}{H_0 - z} \right\| = \sup_{r > \rho_0} \left| \frac{r + \rho_0}{\mathrm{e}^{-i\vartheta} r + E_j - z} \right| \leqslant \sup_{r > \rho_0} \left\{ \frac{r + \rho_0}{r - |E_j - z|} \right\} \leqslant 4, \tag{IV.37}$$

$$\left\| \bar{P}_0^{(2)} \frac{K}{H_0 - z} \right\| \leqslant \sup_{r > 0, \, |t| \geqslant \delta} \left\{ \frac{|t| + r + \rho_0}{|t + \mathrm{e}^{-i\vartheta} r| - |E_j - z|} \right\}. \tag{IV.38}$$

Now, $|t + \mathrm{e}^{-i\vartheta} r|^2 = t^2 + r^2 - 2tr \cos \vartheta \geqslant (1 - \cos \vartheta)(t^2 + r^2)$, and

$$|E_j - z| \leqslant \frac{\rho_0}{2} \leqslant \frac{\delta}{4} \sqrt{1 - \cos \vartheta} \leqslant \frac{|t|}{4} \sqrt{1 - \cos \vartheta}. \tag{IV.39}$$

Thus,

$$\left\| \bar{P}_0^{(2)} \frac{K}{H_0 - z} \right\| \leqslant \frac{\text{const}}{\sqrt{1 - \cos \vartheta}},$$

and hence, for some constant C_0',

$$\left\| \bar{P}_0 \frac{K}{H_0 - z} \right\| \leqslant C_0'. \tag{IV.40}$$

By inserting (IV.40) into the l.s. of (IV.33)–(IV.35), we obtain the bounds

$$\| |H_0 - z|^{-1/2} \, \bar{P}_0 W_l \bar{P}_0 \, |H_0 - z|^{-1/2} \| \leqslant C_0'' \, \| K^{-1/2} W_l K^{-1/2} \|, \tag{IV.41}$$

$$\| |H_0 - z|^{-1/2} \, \bar{P}_0 W_l P_0 \| \leqslant C_0'' \rho_0^{1/2} \, \| K^{-1/2} W_l K^{-1/2} \|, \tag{IV.42}$$

$$\| P_0 W_l \bar{P}_0 \, |H_0 - z|^{-1/2} \| \leqslant C_0'' \rho_0^{1/2} \, \| K^{-1/2} W_l K^{-1/2} \|, \tag{IV.43}$$

for some constant $C_0'' < \infty$. Thus, it remains to be shown that

$$\|K^{-1/2} W_l K^{-1/2}\| \leqslant C_0''' \cdot \rho_0^{-1/2}, \tag{IV.44}$$

for some constant C_0''' because in this case we obtain (IV.33)–(IV.35) from $C_0 := C_0'' \cdot C_0'''$. We demonstrate (IV.44) only for

$$W_{0,q} := \int G_{0,q}(k_1, ..., k_q) \otimes a(k_1) \cdots a(k_q)\, dk_1 \cdots dk_q, \tag{IV.45}$$

where $q=1$ or $q=2$. To this end, we pick a normalized vector $\psi \in \mathscr{H}$ and estimate, using $(k_1, ..., k_q) =: \underline{k}$, ,$\omega(\underline{k}) = \omega(k_1) + \cdots + \omega(k_q)$, and the Pull-Through Formulae (IV.63)–(IV.64),

$$\begin{aligned}
&\|K^{-1/2} W_{0,q} K^{-1/2} \psi\| \\
&\quad = \left\| K^{-1/2} \int (G_{0,q}(\underline{k}) \otimes \mathbf{1}_f)(K + \omega(\underline{k}))^{-1/2} \right. \\
&\qquad \left. \times (\mathbf{1}_{el} \otimes a(k_1) \cdots a(k_q))\, \psi\, dk_1 \cdots dk_q \right\| \\
&\quad \leqslant \left\{ \int \|K^{-1/2}(G_{0,q}(\underline{k}) \otimes \mathbf{1}_f)(K + \omega(\underline{k}))^{-1/2}\, \mathbf{1}_{el} \otimes (H_f + \omega(\underline{k}))^{q/2}\|^2 \right. \\
&\qquad \left. \times \frac{dk_1}{\omega(k_1)} \cdots \frac{dk_q}{\omega(k_q)} \right\}^{1/2} \cdot B_q^{1/2}(\psi),
\end{aligned} \tag{IV.46}$$

where (see (V.51)–(V.52))

$$\begin{aligned}
B_q(\psi) &:= \int \|\mathbf{1}_{el} \otimes (H_f + \omega(\underline{k}))^{-q/2}\, a(k_1) \cdots a(k_q) \psi\|^2\, \omega(k_1)\, dk_1 \cdots \omega(k_q)\, dk_q \\
&\leqslant \|\psi\|^2 = 1.
\end{aligned} \tag{IV.47}$$

Thus

$$\|K^{-1/2} W_{0,q} K^{-1/2}\| \leqslant \left\{ \int M_q(\underline{k})^2 \frac{dk_1}{\omega(k_1)} \cdots \frac{dk_q}{\omega(k_q)} \right\}^{1/2}, \tag{IV.48}$$

where

$$\begin{aligned}
M_q(\underline{k}) &:= \|K^{-1/2}(G_{0,q}(\underline{k}) \otimes \mathbf{1}_f)(K + \omega(\underline{k}))^{-1/2}\, \mathbf{1}_{el} \otimes (H_f + \omega(\underline{k}))^{q/2}\| \\
&= \sup_{r>0} \|(|H_{el} - E_j| + r + \rho_0)^{-1/2}\, G_{0,q}(\underline{k}) \\
&\qquad \times (|H_{el} - E_j| + r + \omega(\underline{k}) + \rho_0)^{-1/2}\, (r + \omega(\underline{k}))^{q/2}\|_{\mathscr{H}_{el}}.
\end{aligned} \tag{IV.49}$$

using the spectral theorem for H_f. In view of (IV.48), it suffices to prove that

$$M_q(\underline{k}) \leqslant \frac{\text{const}}{\rho_0^{1/2}} \prod_{j=1}^{q} (1+\omega(k_j))^{1/2} J(k_j). \tag{IV.50}$$

Now, we distinguish the cases $q=1$ and $q=2$. For $q=1$, we insert (IV.24) into (IV.49) and get

$$\begin{aligned}
M_1(\underline{k}) &\leqslant \sup_{r,t,t'>0} \left\{ \frac{J(k)\,(t+1)^{1/4}\,(t'+1)^{1/4}\,(r+\omega(k))^{1/2}}{[t+r+\rho_0]^{1/2}\,[t'+r+\rho_0+\omega(k)]^{1/2}} \right\} \\
&\leqslant \sup_{r>0} \left\{ \frac{J(k)(r+\rho_0+1)^{1/4}\,(r+\rho_0+\omega(k)+1)^{1/4}\,(r+\omega(k))^{1/2}}{[r+\rho_0]^{1/2}\,[r+\rho_0+\omega(k)]^{1/2}} \right\} \\
&\leqslant \sup_{r>0} \left\{ \frac{J(k)(r+\rho_0+1)^{1/4}\,(r+\rho_0+\omega(k)+1)^{1/4}}{[r+\rho_0]^{1/2}} \right\} \\
&\leqslant (\rho_0+1)^{1/4}\,(\rho_0+\omega(k)+1)^{1/4}\,\rho_0^{-1/2}\,J(k) \\
&\leqslant (\rho_0+1)^{1/2}\,\rho_0^{-1/2}[1+\omega(k)]^{1/4}\,J(k),
\end{aligned} \tag{IV.51}$$

by minimizing over t, t', and $r>0$.

For $q=2$, we have

$$\begin{aligned}
M_2(\underline{k}) &\leqslant \sup_{r>0} \left\{ \frac{J(k_1)\,J(k_2)[r+\omega(\underline{k})]}{[r+\rho_0]^{1/2}\,[r+\rho_0+\omega(\underline{k})]^{1/2}} \right\} \\
&\leqslant [\rho_0+\omega(\underline{k})]^{1/2}\,\rho_0^{-1/2} J(k_1)\,J(k_2) \\
&\leqslant (\rho_0+1)^{1/2}\,\rho_0^{-1/2} \prod_{j=1}^{2} [1+\omega(k_j)]^{1/4}\,J(k_j)
\end{aligned} \tag{IV.52}$$

and, hence, arrive at (IV.50) which, inserted into (IV.48), gives (IV.44) and thus proves the claim (IV.33)–(IV.35). ∎

COROLLARY IV.5. *Let* $z \in D(E_j, \rho_0/2) \equiv \{E_j+\zeta \,|\, |\zeta| \leqslant \rho_0/2\}$. *Then, for* $g>0$ *sufficiently small,* $z \in \sigma[H_g]$ *if and only if* $0 \in \sigma[f_{P_0}(H_g - z)]$, *where*

$$\begin{aligned}
f_{P_0}(H_g - z) = P_{el,j} \otimes \{\chi_{H_f \leqslant \rho_0}(E_j - z + e^{-i\vartheta} H_f)\} + P_0 W_g P_0 \\
- P_0 W_g \overline{P}_0(\overline{P}_0 H_g \overline{P}_0 - z)^{-1}\,\overline{P}_0 W_g P_0.
\end{aligned} \tag{IV.53}$$

Proof. We assume $z \in D(E_j, \rho_0/2)$. We verify the hypotheses (IV.2) and (IV.3) of Theorem IV.1. In order to prove that $\overline{P}_0 H_g \overline{P}_0 - z$ is invertible on $\operatorname{Ran}\{\overline{P}_0\}$, for $|z - E_j| \leqslant \rho_0/2$, we establish absolute convergence of the Neumann series expansion.

$$(\overline{P}_0 H_g \overline{P}_0 - z)^{-1} \overline{P}_0$$
$$= |H_0 - z|^{-1/2} \overline{P}_0 \sum_{n=0}^{\infty} U^* (-|H_0 - z|^{-1/2} \overline{P}_0 W_g \overline{P}_0 |H_0 - z|^{-1/2} U^*)^n$$
$$\times \overline{P}_0 |H_0 - z|^{-1/2}, \tag{IV.54}$$

where $U^* = U^{-1} = |H_0 - z| (H_0 - z)^{-1} \overline{P}_0$ is the unitary on $\mathrm{Ran}\{\overline{P}_0\}$ that results from the Polar decomposition of $H_0 - z$. The series (IV.54) converges in norm, since by Lemma IV.4,

$$\| |H_0 - z|^{-1/2} \overline{P}_0 W_g \overline{P}_0 |H_0 - z|^{-1/2} \| \leqslant \frac{C_0 g}{\rho_0^{1/2}} = C_0 g^{\tau} < \frac{1}{2}, \tag{IV.55}$$

for $g > 0$ sufficiently small, using the definition (IV.26) of $\rho_0 = g^{2-2\tau}$.

Moreover, Lemma IV.4 establishes the boundedness of $\|R_{\overline{P}_0}(z) W_g P_0\|$ and $\|P_0 W_g R_{\overline{P}_0}(z)\|$, as well. Thus the hypotheses of Theorem IV.1 are satisfied. ∎

IV.4. *The Spectrum of the Feshbach Hamiltonian*, $f_{P_0}(H_g(\theta) - z)$

As a consequence of Corollary IV.5 in the preceding section, $\sigma[H_g] \cap D(E_j, \rho_0/2)$ is given by those $z \in D(E_j, \rho_0/2)$ for which $0 \in \sigma[f_{P_0}(H_g - z)]$, where

$$f_{P_0}(H_g - z) = P_{el,j} \otimes \{\chi_{H_f \leqslant \rho_0}(E_j - z + e^{-i\vartheta} H_f)\} + P_0 W_g P_0$$
$$- P_0 W_g \overline{P}_0 (\overline{P}_0 H_g \overline{P}_0 - z)^{-1} \overline{P}_0 W_g P_0. \tag{IV.56}$$

Our goal is to show that if $z \in D(E_j, \rho_0/2)$ and $0 \in \sigma[f_{P_0}(H_g - z)]$ then

$$z \in E_j - \Delta E_j(g) + e^{-i\vartheta} R_+ + O(g^{2+(\beta/(2+\beta))}), \tag{IV.57}$$

where $\mathrm{Im}\{\Delta E_j(g)\} = -\gamma_j$. (The definition of γ_j is given in Theorem IV.2). This will imply Theorem IV.3. The proof of (IV.57) is accomplished in a sequence of lemmata.

Lemma IV.6.

$$\|P_0 W_g \overline{P}_0 [(\overline{P}_0 H_g \overline{P}_0 - z)^{-1} - (\overline{P}_0 H_0 \overline{P}_0 - z)^{-1}] \overline{P}_0 W_g P_0\| = O(g^{2+\tau}). \tag{IV.58}$$

Proof. By the 2^{nd} resolvent equation, the left side of (IV.58) is bounded by

$$\sum_{l_1, l_2, l_3=1}^{2} g^{l_1+l_2+l_3} \| P_0 W_{l_1} \bar{P}_0 \, |H_0 - z|^{-1/2} \|$$
$$\times \| |H_0 - z|^{-1/2} \bar{P}_0 W_{l_2} \bar{P}_0 \, |H_0 - z|^{-1/2} \| \; \| |H_0 - z|^{-1/2} \bar{P}_0 W_{l_3} P_0 \|$$
$$\times \| |H_0 - z|^{1/2} \bar{P}_0 (\bar{P}_0 H_g \bar{P}_0 - z)^{-1} \bar{P}_0 \, |H_0 - z|^{1/2} \|$$
$$\leqslant 16 C_0^3 g^3 \rho_0^{-1/2}, \tag{IV.59}$$

using Lemma IV.4 and (IV.54), (IV.55). ∎

Lemma IV.7. *Let*

$$\mathrm{Rem}_1(g, \rho_0) := P_0 W_g \bar{P}_0 (\bar{P}_0 H_g \bar{P}_0 - z)^{-1} \bar{P}_0 W_g P_0 - g^2 P_0 W_1 \left(\frac{\bar{P}_0}{H_0 - z} \right) W_1 P_0. \tag{IV.60}$$

Then

$$\| \mathrm{Rem}_1(g, \rho_0) \| = O(g^{2+\tau}). \tag{IV.61}$$

Proof. By Lemma IV.6, it suffices to show that

$$\left\| \sum_{l+l' \geqslant 3} g^{l+l'} P_0 W_l \left(\frac{\bar{P}_0}{H_0 - z} \right) W_{l'} P_0 \right\| \leqslant O(g^3), \tag{IV.62}$$

which follows directly from (IV.34), (IV.35) in Lemma IV.4. ∎

Next, we use the following Pull-Through formula.

Lemma IV.8 (Pull-Through Formulae). *Let $F: \mathbb{R}_+ \to \mathbb{C}$ be a Borel function with $F[r] = O(r+1)$. Then $F[H_f]$, defined by the spectral theorem for H_f, is defined on the domain of H_f, and it obeys the following intertwining relations*:

$$F[H_f] \, a^\dagger(k) = a^\dagger(k) \, F[H_f + \omega(k)], \tag{IV.63}$$

$$a(k) \, F[H_f] = F[H_f + \omega(k)] \, a(k). \tag{IV.64}$$

Equations (IV.63) and (IV.64) follow immediately from

$$F[H_f] \prod_{i=1}^{N} a^\dagger(k_i) \Omega = F\left[\sum_{i=1}^{N} \omega(k_i) \right] \prod_{i=1}^{N} a^\dagger(k_i) \Omega. \tag{IV.65}$$

We apply the Pull-Through formulae (IV.63)–(IV.64) and rewrite

$$
\begin{aligned}
&P_0 W_1 \left(\frac{\bar{P}_0}{H_0 - z}\right) W_1 P_0 \\
&\quad = Q + \int P_0 [G_{1,0}(k) \otimes a^\dagger(k)\, a^\dagger(k')] \\
&\quad\quad \times \left[\frac{\bar{P}_0(\omega(k'))}{H_0 + e^{-i\vartheta}\omega(k') - z}\right] [G_{1,0}(k') \otimes \mathbf{1}_f]\, P_0\, dk\, dk' \\
&\quad\quad + \int P_0 [G_{0,1}(k) \otimes \mathbf{1}_f] \left[\frac{\bar{P}_0(\omega(k))}{H_0 + e^{-i\vartheta}\omega(k) - z}\right] \\
&\quad\quad \times [G_{0,1}(k') \otimes a(k)\, a(k')]\, P_0\, dk\, dk' \\
&\quad\quad + \int P_0 [G_{1,0}(k) \otimes a^\dagger(k)] \left[\frac{\bar{P}_0}{H_0 - z}\right] [G_{0,1}(k') \otimes a(k')]\, P_0\, dk\, dk' \\
&\quad\quad + \int P_0 [G_{0,1}(k) \otimes a^\dagger(k')] \left[\frac{\bar{P}_0(\omega(k) + \omega(k'))}{H_0 + e^{-i\vartheta}(\omega(k) + \omega(k')) - z}\right] \\
&\quad\quad \times [G_{1,0}(k') \otimes a(k)]\, P_0\, dk\, dk',
\end{aligned} \tag{IV.66}
$$

where

$$
Q := \int P_0 [G_{0,1}(k) \otimes \mathbf{1}_f] \left[\frac{\bar{P}_0(\omega(k))}{H_0 + e^{-i\vartheta}\omega(k) - z}\right] [G_{1,0}(k) \otimes \mathbf{1}_f]\, P_0\, dk, \tag{IV.67}
$$

and $\bar{P}_0(\omega) := \mathbf{1} - P_0(\omega)$ with $P_0(\omega) := P_{el,j} \otimes \chi_{H_f + \omega < \rho_0}$.

LEMMA IV.9. *For ρ_0 as in* (IV.26)–(IV.27),

$$
\mathrm{Rem}_2(g, \rho_0) := g^2 P_0 W_1 \left[\frac{\bar{P}_0}{H_0 - z}\right] W_1 P_0 - g^2 Q \tag{IV.68}
$$

is a bounded operator from $P_0(\mathscr{H}_{el} \otimes \mathscr{F})$ to itself, with

$$
\|\mathrm{Rem}_2(g, \rho_0)\| = O(g^{2 + 2\beta(1-\tau)}). \tag{IV.69}
$$

Proof. The operator $\mathrm{Rem}_2(g, \rho_0)$ is given by the last four terms on the right side of (IV.66). Obviously all these terms map the subspace $P_0(\mathscr{H}_{el} \otimes \mathscr{F})$ to itself. We estimate the norms of all four terms separately. The estimates on the norms of the operators proportional to $a^\dagger(k)\, a^\dagger(k')$ and to $a(k)\, a(k')$, respectively, are analogous; we only examine the one proportional to $a(k)\,a(k')$. Condition (IV.24) states that

$$\|(|H_{el}-E_j|+\mathbf{1})^{-1/4}\,G_{m,n}(k)(|H_{el}-E_j|+\mathbf{1})^{-1/4}\|\leqslant J(k),$$

for $m+n=1$. Thus, for an arbitrary vector ψ,

$$\begin{aligned}&\left\|\int P_0[G_{0,1}(k)\otimes\mathbf{1}_f]\left[\frac{\bar{P}_0(\omega(k))}{H_0+\mathrm{e}^{-i\vartheta}\omega(k)-z}\right]\right.\\&\qquad\left.\times[G_{0,1}(k')\otimes a(k)\,a(k')]\,P_0\psi\,dk\,dk'\right\|\\&\quad\leqslant\int J(k)\,J(k')\left\|\frac{\bar{P}_0(\omega(k))(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes\mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k))-z}\right\|\\&\qquad\cdot\|\mathbf{1}_{el}\otimes a(k)\,a(k')\,\chi_{H_f<\rho_0}\psi\|\,dk\,dk'.\end{aligned}\tag{IV.70}$$

We recall from Eqs. (IV.30) and (IV.31) that

$$\bar{P}_0(\omega)=\bar{P}_0^{(1)}(\omega)+\bar{P}_0^{(2)},$$

where

$$\bar{P}_0^{(1)}(\omega)=P_{el,j}\otimes\chi_{H_f+\omega\geqslant\rho_0},\tag{IV.71}$$

$$\bar{P}_0^{(2)}=\bar{P}_{el,j}\otimes\mathbf{1}_f.\tag{IV.72}$$

Since $|E_j-z|\leqslant\rho_0/2$, we have that

$$\begin{aligned}&\left\|\frac{\bar{P}_0^{(1)}(\omega(k))(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes\mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k)))-z}\right\|\\&\quad=\left\|\frac{\chi_{H_f+\omega(k)\geqslant\rho_0}}{\mathrm{e}^{-i\vartheta}(H_f+\omega(k))+E_j-z}\right\|_{\mathscr{F}}\\&\quad\leqslant\sup_{r\geqslant\max\{0,\,\rho_0-\omega(k)\}}\left\{\frac{1}{r+\omega(k)-|E_j-z|}\right\}\leqslant\frac{4}{\omega(k)+\rho_0}.\end{aligned}\tag{IV.73}$$

Furthermore, using that $|E_j-z|\leqslant\rho_0/2\leqslant\delta\sqrt{1-\cos\vartheta}/4$, by (IV.27), we find that

$$\left\| \frac{\bar{P}_0^{(2)}(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes \mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k))-z} \right\|$$

$$\leqslant \sup_{t\geqslant\delta,\, r>0} \left| \frac{4(t+1)^{1/2}}{4\,|t+\mathrm{e}^{-i\vartheta}(r+\omega(k))|-\delta\sqrt{1-\cos\vartheta}} \right|$$

$$\leqslant \sup_{t\geqslant\delta} \left\{ \frac{8\sqrt{2}\,[2t+2+\omega(k)]^{1/2}}{3\sqrt{1-\cos\vartheta}\,[t+\omega(k)]} \right\} \leqslant \frac{4\,[1+\delta]^{1/2}}{\sqrt{1-\cos\vartheta}\,[\delta+\omega(k)]}. \tag{IV.74}$$

Combining (IV.73) and (IV.74), we obtain the bound

$$\left\| \frac{\bar{P}_0(\omega(k))(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes \mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k))-z} \right\| \leqslant \frac{C_1}{\omega(k)+\rho_0}, \tag{IV.75}$$

where $C_1 := 4(1+\delta)^{1/2}\,(1-\cos\vartheta)^{-1/2}$. We may now complete our bound on the right side of (IV.70): Using (IV.75) we find that

$$\begin{aligned} \text{R.S. of (IV.70)} &\leqslant C_1 \int \frac{J(k)\,J(k')}{\omega(k)+\rho_0}\,\|\mathbf{1}_{el}\otimes a(k)\,a(k')\,\chi_{H_f<\rho_0}\psi\|\,dk\,dk' \\ &\leqslant C_1\,\rho_0^{\beta-1}\left(\int_{|k|\leqslant\rho_0} \frac{J(k)^2}{\omega(k)^{1+\beta}}\,dk\right)\|\mathbf{1}_{el}\otimes H_f\,\chi_{H_f<\rho_0}\psi\| \\ &\leqslant C_1\,\Lambda_{1+\beta}^2\rho_0^{\beta}\,\|\psi\|. \end{aligned} \tag{IV.76}$$

The estimates on the remaining two terms in $\mathrm{Rem}_2(g,\rho_0)$ are carried out in the same fashion; the resulting bounds are similar to (IV.76). This completes the proof of Lemma IV.9. ∎

Lemma IV.10. *Let* $\mathrm{Rem}_3(g,\rho_0) := g^2 P_0 W_2 P_0$. *Then*

$$\|\mathrm{Rem}_3(g,\rho_0)\| \leqslant 3g^{2+2(1+\beta)(1-\tau)}\,\Lambda_{1+\beta}^2. \tag{IV.77}$$

Proof. By Eq. (IV.18), $P_0 W_2 P_0$ is the sum of three terms. We estimate them separately, but all three estimates have the same structure. As an example we estimate the term proportional to $G_{1,1}(k,k')\otimes a^{\dagger}(k)\,a(k')$: For arbitary vectors ϕ and ψ in $H_{el}\otimes F$,

$$\begin{aligned} &\left| \int \langle P_0\phi\,|\,G_{11}(k,k')\otimes a^{\dagger}(k)\,a(k')\,P_0\psi\rangle\,dk\,dk' \right| \\ &\qquad\leqslant \int J(k)\,J(k')\,\|a(k)\,P_0\phi\|\cdot\|a(k')\,P_0\psi\|\,dk\,dk' \\ &\qquad\leqslant \rho_0^{\beta}\Lambda_{1+\beta}^2\cdot\|\mathbf{1}_{el}\otimes H_f^{1/2}P_0\phi\|\cdot\|\mathbf{1}_{el}\otimes H_f^{1/2}P_0\psi\|, \\ &\qquad\leqslant \rho_0^{1+\beta}\Lambda_{1+\beta}^2\cdot\|\phi\|\cdot\|\psi\|. \quad ∎ \end{aligned} \tag{IV.78}$$

The results in Lemmata IV.6 through IV.10 may be summarized as follows. If $f_{P_0}(H_g - z)$ denotes the effective Hamiltonian (Feshbach map of $H_g - z$) at a photon energy scale ρ_0, as defined in Eq. (IV.53), then

$$f_{P_0}(H_g - z) = P_0 \mathbf{1}_{el} \otimes (E_j - z + \mathrm{e}^{-i\vartheta} H_f)\, P_0 \\ + g P_0 W_1 P_0 + g^2 Q + \sum_{\mu=1}^{3} \mathrm{Rem}_\mu(g, \rho_0), \tag{IV.79}$$

with

$$\|\mathrm{Rem}_1(g, \rho_0)\| = O(g^{2+\tau}), \tag{IV.80}$$

by Lemma IV.7, and

$$\|\mathrm{Rem}_2(g, \rho_0)\| = O(g^{2+2\beta(1-\tau)}), \tag{IV.81}$$

by Lemma IV.9, where $\mathrm{Rem}_1(g, \rho_0)$ is given by (IV.60), and $\mathrm{Rem}_2(g, \rho_0)$ is given by (IV.68). Furthermore,

$$\|\mathrm{Rem}_3(g, \rho_0)\| = O(g^{2+2(1+\beta)(1-\tau)})\, \Lambda^2_{1+\beta}, \tag{IV.82}$$

by Lemma IV.10. All O-symbols in (IV.81)–(IV.82) represent explicitly computable constants which possibly depend on ϑ, δ, and β.

Our next task is to analyze the term Q on the right side of (IV.79). We observe that, by (IV.67),

$$g^2 Q = g^2 (Z_j^{\mathrm{d}} + Z_j^{\mathrm{od}}) \otimes \chi_{H_f < \rho_0} + \mathrm{Rem}_4(g, \rho_0) + \mathrm{Rem}_5(g, \rho_0), \tag{IV.83}$$

where

$$Z_j^{\mathrm{d}} := \int P_{el,j} G_{0,1}(k)\, P_{el,j} G_{1,0}(k)\, P_{el,j} \frac{dk}{\mathrm{e}^{-i\vartheta}\omega(k)}, \tag{IV.84}$$

$$Z_j^{\mathrm{od}} := \int P_{el,j} G^{(\theta)}_{0,1}(k) [\bar{P}_{el,j} H_{el} - E_j + \mathrm{e}^{-i\vartheta}\omega(k)]^{-1}\, G^{(\theta)}_{1,0}(k)\, P_{el,j}\, dk, \tag{IV.85}$$

and, furthermore,

$$\mathrm{Rem}_4(g, \rho_0) := g^2 \int P_{el,j} (G_{0,1}(k) \otimes \mathbf{1}_f) \\ \times [\bar{P}_{el,j}(H_{el} \otimes \mathbf{1}_f) + \mathrm{e}^{-i\vartheta}(H_f + \omega(k)) - z]^{-1} \\ \times \left[\frac{\mathrm{e}^{-i\vartheta} \mathbf{1}_{el} \otimes H_f \chi_{H_f < \rho_0} + E_j - z}{\bar{P}_{el,j} H_{el} \otimes \mathbf{1}_f - E_j + \mathrm{e}^{-i\vartheta}\omega(k)} \right] \\ \times (G_{1,0}(k) \otimes \mathbf{1}_f)\, P_{el,j} \otimes \chi_{H_f < \rho_0}\, dk, \tag{IV.86}$$

$$\mathrm{Rem}_5(g,\rho_0) := g^2 \left\{ \int \mathcal{G}(k) \otimes [\mathrm{e}^{-i\vartheta}(H_f + \omega(k)) + E_j - z]^{-1} \right.$$

$$\left. \times \chi_{H_f+\omega(k) \geqslant \rho_0}\, dk - Z_j^{\mathrm{d}} \right\} \mathbf{1}_{el} \otimes \chi_{H_f < \rho_0}, \tag{IV.87}$$

with

$$\mathcal{G}(k) := P_{el,j} G_{0,1}(k)\, P_{el,j} G_{1,0}(k)\, P_{el,j}. \tag{IV.88}$$

LEMMA IV.11. *For* $|z - E_j| \leqslant \rho_0/2$,

$$\|\mathrm{Rem}_4(g,\rho_0)\| = O(g^{4-2\tau}). \tag{IV.89}$$

Proof. Since

$$\left\| \frac{(|\overline{P}_{el,j} H_{el} - E_0| + \mathbf{1}_{el})^{1/4}}{\overline{P}_{el,j} H_{el} \otimes \mathbf{1}_f + \mathrm{e}^{-i\vartheta} \mathbf{1}_{el} \otimes (H_f + \omega(k)) - z} \right\| \leqslant C_2 \delta^{-1},$$

uniformly in $|z - E_j| \leqslant \rho_0/2$, we may bound the norm of $\mathrm{Rem}_4(g,\rho_0)$ by

$$C_2^2 \delta^{-2}\, \|\mathrm{e}^{-i\vartheta} \mathbf{1}_{el} \otimes H_f \chi_{H_f < \rho_0} + E_j - z\| \int J(k)^2\, dk \leqslant \left(\frac{3C_2^2 \Lambda_{1+\beta}^2}{\delta^2} \right) \rho_0. \quad \blacksquare$$

Next, we note that in the first term on the right side of (IV.85) we may rotate the k-integration to the subspace $e^{i\vartheta}\mathbb{R}^3$, so as to find that

$$Z_j^{\mathrm{od}} = \int P_{el,j} G_{0,1}^{(\theta)}(k) [\overline{P}_{el,j} H_{el} - E_j + \mathrm{e}^{-i\vartheta} \omega(k)]^{-1}\, G_{1,0}^{(\theta)}(k)\, P_{el,j}\, dk$$

$$= \int P_{el,j} G_{0,1}^{(0)}(k) [\overline{P}_{el,j} H_{el} - E_j + \omega(k) - i0]^{-1}\, G_{1,0}^{(0)}(k)\, P_{el,j}\, dk, \tag{IV.90}$$

and hence

$$\mathrm{Im}\{Z_j^{\mathrm{od}}\} = -A_j, \tag{IV.91}$$

where the $N_j \times N_j$ matrix A_j has been defined in Eq. (IV.19). Here we use the notation

$$\mathrm{Re}\{Z\} = \frac{1}{2}(Z + Z^*) \qquad \text{and} \qquad \mathrm{Im}\{Z\} = \frac{1}{2i}(Z - Z^*), \tag{IV.92}$$

where Z is a matrix.

Next, we deal with the third term in Q on the right side of Eq. (IV.83). Our bound is contained in the following lemma.

LEMMA IV.12. *For* $|z - E_j| \leqslant \rho_0/2$,

$$\|\mathrm{Rem}_5(g, \rho_0)\| \leqslant 7\Lambda^2_{1+\beta} g^{2+2\beta(1-\tau)}. \tag{IV.93}$$

Proof. First, we observe that

$$\int \mathscr{G}(k) \otimes \left\{ [\mathrm{e}^{-i\vartheta}(H_f + \omega(k)) + E_j - z]^{-1} \chi_{H_f+\omega(k) \geqslant \rho_0} - \frac{1}{\mathrm{e}^{-i\vartheta}\omega(k)} \right\} dk\, \chi_{H_f < \rho_0}$$

$$= \left\{ \int \mathscr{G}(k) \otimes \frac{-(\mathrm{e}^{-i\vartheta} H_f + E_j - z)}{(\mathrm{e}^{-i\vartheta}(H_f + \omega(k)) + E_j - z)\, \mathrm{e}^{-i\vartheta}\omega(k)} \chi_{H_f+\omega(k) \geqslant \rho_0}\, dk \right.$$

$$\left. + \int \mathscr{G}(k) \otimes [\chi_{H_f+\omega(k) \geqslant \rho_0} - 1] \frac{dk}{\mathrm{e}^{-i\vartheta}\omega(k)} \right\} \chi_{H_f < \rho_0} \tag{IV.94}$$

Using (IV.88), (IV.24) and the definition of $P_{el,j}$, we find that the norm of the right side is bounded by

$$\int J(k)^2 \left\{ \frac{6\rho_0}{[\rho_0 + \omega(k)]\, \omega(k)} + \frac{\chi_{\omega(k) < \rho_0}}{\omega(k)} \right\} dk \leqslant 7\delta_j^2 \rho_0^\beta \Lambda^2_{1+\beta}. \quad \blacksquare$$

IV.4.1. *Proof of Theorem* IV.3. To begin the proof of Theorem IV.3, we recall from Corollary IV.5 in Section IV.3 that $\sigma[H_g] \cap D(E_j, \rho_0/2)$ is given by those $z \in D(E_j, \rho_0/2)$ for which $0 \in \sigma[f_{P_0}(H_g - z)]$, where

$$f_{P_0}(H_g - z) = P_{el,j} \otimes \{\chi_{H_f < \rho_0}(E_j - z + \mathrm{e}^{-i\vartheta} H_f)\} + P_0 W_g P_0$$
$$- P_0 W_g \bar{P}_0 (\bar{P}_0 H_g \bar{P}_0 - z)^{-1} \bar{P}_0 W_g P_0. \tag{IV.95}$$

We prove Theorem IV.3 in two steps. First, we show that

$$\{z \mid 0 \in \sigma[f_{P_0}(H_g - z)] \cap D(E_j, \rho_0/2)\} \subseteq E_j - \Delta E_j(g)$$
$$+ \mathrm{e}^{-i\vartheta} R_+ + O(g^{2+(\beta/(2+\beta))}), \tag{IV.96}$$

where $\mathrm{Im}\{\Delta E_j(g)\} = -\gamma_j$. (The definition of γ_j is given in Theorem IV.2). This implies that

$$\mathscr{I}_j^{(1)} := \{z \in D(E_j, \rho_0/2) \mid \mathrm{Im}\{z\} > -a_j g^2 + C_4 g^{2+(\beta/(2+\beta))}\}$$
$$\subseteq \rho[H_g], \tag{IV.97}$$

where $\rho[H_g]$ is the resolvent set of $H_g = H_g(\theta)$. Secondly, we show that

$$\mathscr{I}_j^{(2)} := \mathscr{I}_j \backslash \mathscr{I}_j^{(1)} \subseteq \rho[H_g], \tag{IV.98}$$

where $\mathscr{I}_j$ was defined in (IV.22) to be the set

$$\mathscr{I}_j := [\tfrac{2}{3}E_{j-1} + \tfrac{1}{3}E_j, \tfrac{1}{3}E_j + \tfrac{2}{3}E_{j+1}] - i\gamma_j + O(g^{2+(\beta/(2+\beta))}) + i\mathbb{R}_+.$$

Theorem IV.3 then follows by putting together (IV.97) and (IV.98).

In order to establish (IV.96), we first remark that Z_j^{d} in (IV.84) is hermitian, i.e., $Z_j^{\mathrm{d}} = \mathrm{Re}\{Z_j^{\mathrm{d}}\}$. We find this by deforming the k-integration on the right side of (IV.84) into the complex domain. Returning to expression (IV.79) for the effective Hamiltonian $f_{P_0}[H_g - z]$ at photon energy scale ρ_0 and using (IV.81)–(IV.82), Lemma IV.11 and Lemma IV.12, we conclude that

$$f_{P_0}(H_g - z) = P_0(E_j - z + g^2[Z^{\mathrm{d}} + Z^{\mathrm{od}}] \otimes \mathbf{1}_f + \mathrm{e}^{-i\vartheta}\mathbf{1}_{el} \otimes H_f) P_0$$
$$+ gP_0 W_1 P_0 + \sum_{\mu=1}^{5} \mathrm{Rem}_\mu(g, \rho_0), \tag{IV.99}$$

where $g^2 Z_j^{\mathrm{d}}$ and $g^2 Z_j^{\mathrm{od}}$ are $O(g^2)$, and

$$\left\| \sum_{\mu=1}^{5} \mathrm{Rem}_\mu(g, \rho_0) \right\| = O(g^{4-2\tau} + g^{2+\tau} + g^{2+2(1+\beta)(1-\tau)}). \tag{IV.100}$$

Furthermore, it is easy to show that

$$\| gP_0 W_1 P_0 \| \leqslant C_3 g \rho_0^{(1+\beta)/2} = O(g^{1+(1-\tau)(1+\beta)}). \tag{IV.101}$$

Given $\beta > 0$ such that $\Lambda_{1+\beta} < \infty$, we choose

$$\tau := \frac{\beta}{2+\beta} \qquad \text{and hence} \qquad \rho_0 = g^{2-(2\beta/2+\beta)} \tag{IV.102}$$

in (IV.26). The coupling constant $g \geqslant 0$ has to be so small that both Condition (IV.27) and (IV.55) are satisfied. Adding up the error terms in (IV.100) and (IV.101), we obtain

$$\left\| gP_0 W_1 P_0 + \sum_{\mu=1}^{5} \mathrm{Rem}_\mu(g, \rho_0) \right\| = O(g^{2+(\beta/(2+\beta))}). \tag{IV.103}$$

Assuming that (IV.19) in Theorem IV.2(a) is valid, i.e.,

$$g^2 A_j \geqslant g^2 a_j \mathbf{1} > 0, \tag{IV.104}$$

we conclude from (IV.99), (IV.103) and (IV.104) that, for g small enough and a sufficiently large constant, $C_4 \equiv C_4(\vartheta, \delta, \beta)$, the operator $f_{P_0}(H_g - z)$ has a bounded inverse whenever

$$|z - E_j| \leqslant \frac{\rho_0}{2} \qquad \text{and} \qquad \operatorname{Im}\{z\} > -a_j g^2 + C_4 g^{2+(\beta/(2+\beta))}, \tag{IV.105}$$

which is equivalent to (IV.97).

To prove (IV.98), we first note that all $z \in \mathscr{I}_j^{(2)}$ obey $\pi + \vartheta/2 \geqslant \arg\{z - E_j\} \geqslant -\vartheta/2$, provided that

$$\gamma_j = a_j g^2 \leqslant \frac{\rho_0}{6} \sin(\vartheta/2). \tag{IV.106}$$

Thus, for any $z \in \mathscr{I}_j^{(2)}$,

$$\begin{aligned} \left\| \frac{K(P_{el,j} \otimes \mathbf{1})}{H_0 - z} \right\| &= \sup_{r>0} \left\{ \frac{r + \rho_0}{|E_j - z + \mathrm{e}^{-i\vartheta} r|} \right\} \\ &\leqslant \sup_{r>0} \left\{ \frac{r + \rho_0}{|(\rho_0/2) + \mathrm{e}^{-i\vartheta/2} r|} \right\} \leqslant \frac{\sqrt{8}}{\sqrt{1 - \cos(\vartheta/2)}}, \end{aligned} \tag{IV.107}$$

where K is defined in (IV.36). On the other hand, for any $z \in \mathscr{I}_j^{(2)}$,

$$\begin{aligned} \left\| \frac{K(\overline{P}_{el,j} \otimes \mathbf{1})}{H_0 - z} \right\| &= \sup_{r>0,\, |t| \geqslant \delta} \left\{ \frac{|t| + r + \rho_0}{|t + E_j - z + \mathrm{e}^{-i\vartheta} r|} \right\} \\ &\leqslant \frac{16}{\sqrt{1 - \cos\vartheta}}, \end{aligned} \tag{IV.108}$$

as is easily verified by separately examining the cases $r > |t|/6$ and $r \leqslant |t|/6$, using (IV.106) and the fact that $\sin\vartheta \geqslant 1 - \cos\vartheta$. From (IV.107) and (IV.108) we conclude that

$$\sup_{z \in \mathscr{I}_j^{(2)}} \left\| \frac{K}{H_0 - z} \right\| \leqslant \frac{16}{\sqrt{1 - \cos\vartheta}}. \tag{IV.109}$$

So, finally, we obtain the invertibility of $H_g - z$ and thus (IV.98) from an expansion in a Neumann series as in (IV.54). This series is norm convergent since, by (IV.109) and (IV.44),

$$\| |H_0 - z|^{-1/2} W_g |H_0 - z|^{-1/2} \| \leqslant \left\| \frac{K}{H_0 - z} \right\| \cdot \| K^{-1/2} W_g K^{-1/2} \|$$
$$\leqslant \frac{16\, C_0 g \rho_0^{-1/2}}{\sqrt{1 - \cos \vartheta}} = \frac{16 C_0 g^{\beta/(2+\beta)}}{\sqrt{1 - \cos \vartheta}} < 1, \tag{IV.110}$$

for $g > 0$ sufficiently small. This completes the proof of Theorem IV.3. ▮

V. EFFECTIVE HAMILTONIAN AND RENORMALIZATION GROUP

The purpose of this chapter is to analyze the flow of effective Hamiltonians under renormalization transformations, lowering the photon energy scale. Our analysis exhibits the infrared asymptotic freedom of QED as described by the Hamiltonian H_g, defined in (I.22). As an application of our methods, we shall prove the existence of a ground state, with an estimate on its multiplicity, and the existence of resonances as eigenvalues of the complex dilated Hamiltonian, $H_g(\theta)$ in (IV.21)).

This chapter is organized as follows. In Section V.1 we describe the general strategy of our renormalization group analysis. In Section V.2 we eliminate the electron degrees of freedom with the help of the isospectral Feshbach map. In Section V.3 we outline the proof of the fact that the renormalization map, applied to the effective model derived in Section V.2, is a contraction. From this we obtain information on the spectrum of $H_g(\theta)$. As we demonstrate in Section V.5, the fixed point of the renormalization map gives rise to an eigenvalue of $H_g(\theta)$, the resonance sought for.

V.1. *The General Strategy of the Renormalization Group Construction*

In this section, we describe the key ideas underlying our renormalization group construction of resonances. Recall that the latter are defined as complex eigenvalues of the Hamiltonian $H_g(\theta)$ considered in Eq. (IV.21). Hypotheses 2 and 3 of Section I.3 are required throughout this chapter. Technical estimates will be supplied in subsequent sections.

V.1.1. *Passing from a single operator to a Banach space op operators.* We start from the effective Hamiltonian

$$e^{-i\vartheta} [f_{P_0}(H_g(\theta) - z) - (E_j - z) P_0], \tag{V.1}$$

defined in (IV.53). Recall that

$$P_0 = P_{el,j} \otimes \chi_{H_f < \rho_0}, \tag{V.2}$$

where $P_{el,j}$ is the orthogonal projection onto the eigenspace of the atomic Hamiltonian H_{el} corresponding to the eigenvalue E_j, and $\chi_{H_f<\rho_0}$ is the spectral projection of H_f onto the subspace of vectors in Fock space with field energy $<\rho_0$; furthermore z belongs to the disk $D(E_j, \rho_0/2)$, i.e., $|z-E_j| \leqslant \rho_0/2$. For simplicity we assume that the eigenvalue E_j is *simple*, so that $P_{el,j}$ is a one-dimensional projection. Then we can view the operator in (V.1) as acting on the Hilbert space $\chi_{H_f<\rho_0}\mathscr{F}$.

First, we rescale the photon momenta by means of the unitary dilatation $U_{\rho_0} := U_f(-\ln \rho_0)$,

$$k \to \rho_0 k, \tag{V.3}$$

passing from the operator in (V.1) to a unitarily equivalent Hamiltonian, $\rho_0 H_{\mathrm{eff}}[\zeta]$, which is defined by

$$H_{\mathrm{eff}}[\zeta] = \frac{\mathrm{e}^{-i\vartheta}}{\rho_0} U_{\rho_0}[f_{P_0}(H_g(\theta)-z) - (E_j-\zeta)\,\chi_{H_f<\rho_0}]\,U^*_{\rho_0}, \tag{V.4}$$

for all $\zeta \in D(E_j, \rho_0/2)$, on the Hilbert space

$$\mathscr{H}_{\mathrm{red}} := \chi_{H_f<1}\mathscr{F} \equiv \mathrm{Ran}\,\chi_{H_f<1}. \tag{V.5}$$

Following the photon momenta, we map the spectral parameter, z, as well. We introduce the bijection

$$Z_{(0)}\colon D(E_j, \rho_0/2) \to D_{1/2}, \qquad \zeta \mapsto \frac{\mathrm{e}^{i\vartheta}}{\rho_0}(\zeta - E_j) \tag{V.6}$$

(see Fig. V.1), and we define

$$H_{(0)}[Z_{(0)}(\zeta)] := H_{\mathrm{eff}}[\zeta], \tag{V.7}$$

for all $\zeta \in D(E_j, \rho_0/2)$. Composing these two operations, we explicitly have

$$H_{(0)}[z] = \frac{\mathrm{e}^{-i\vartheta}}{\rho_0} U_{\rho_0}[f_{P_0}(H_g(\theta) - Z_{(0)}^{-1}[z]) - (E_j - Z_{(0)}^{-1}[z])\,\chi_{H_f<\rho_0}]\,U^*_{\rho_0}, \tag{V.8}$$

on $\mathscr{H}_{\mathrm{red}}$, for all $z \in D_{1/2} := \{|z| \leqslant 1/2\}$.

After rescaling of the photon momenta and the spectral parameter, we expand the resolvent $\bar{P}_0(\bar{P}_0 H_g(\theta)\,\bar{P}_0 - \zeta)^{-1}\,\bar{P}_0$, where $\bar{P}_0 = \mathbf{1} - P_0$, entering $f_{P_0}(H_g(\theta)-\zeta)$ in (V.8), in $\bar{P}_0 W_g \bar{P}_0$. Using a straightforward generalization of

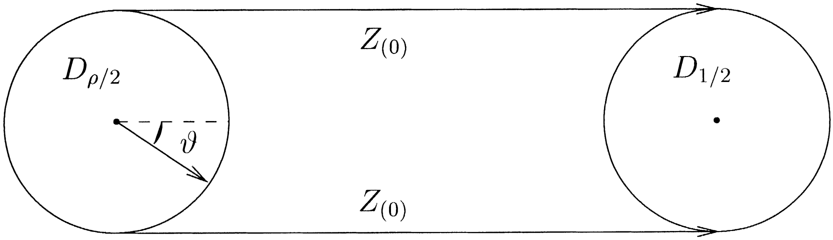


FIG. V.1. First rescaling of the spectral parameter.

Wick's theorem, we show that the operator $H_{(0)}[z]$ can be represented in the form

$$H_{(0)}[z] := \chi_{H_f<1}\left(E_{(0)}[z]\cdot\mathbf{1} + T_{(0)}[z; H_f] + \sum_{M+N\geqslant 1} W^{(0)}_{M,N}\right)\chi_{H_f<1} \tag{V.9}$$

where $E_{(0)}[z] \in \mathbb{C}$ is a number, $T_{(0)}[z; H_f]$ is a spectral function of H_f, and $W^{(0)}_{M,N}$ are "Wick monomials" of the form

$$W^{(0)}_{M,N}[z] = \int dk^{(M)}\, d\tilde{k}^{(N)}\, a^{\dagger}(k^{(M)})\, w^{(0)}_{M,N}[H_f; z; k^{(M)}, \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}), \tag{V.10}$$

for $M+N\geqslant 1$. Here, we use the notation introduced in Section I.3, i.e.,

$$k^{(m)} := (k_1, ..., k_m) \in \mathbb{R}^{3m}, \qquad dk^{(m)} := \prod_{i=1}^{m} d^3 k_i, \tag{V.11}$$

$$a^{\dagger}(k^{(m)}) := \prod_{i=1}^{m} a^{\dagger}(k_i), \qquad \omega(k^{(m)}) := \sum_{i=1}^{m} \omega(k_i). \tag{V.12}$$

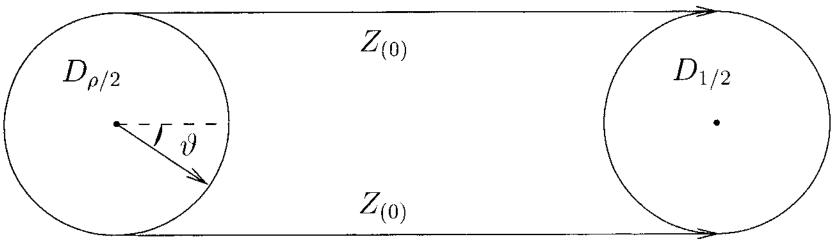
In Subsection V.2.2, we will show that, for each $z \in D_{1/2}$, $H_{(0)}[z]$ belongs to a certain *Banach space*, $\mathscr{W}'_{\Delta}$, of Hamiltonians on $\mathscr{H}_{[\text{red}]} = \chi_{H_f<1}\mathscr{F}$, which we define below.

The Banach space $\mathscr{W}'_{\Delta}$, defined as

$$\mathscr{W}'_{\Delta} := \mathbb{C} \oplus \mathscr{T} \oplus \bigoplus_{M+N\geqslant 1} \mathscr{W}_{\Delta}(M, N), \tag{V.13}$$

depends on three parameters $0<\rho<1/16$, $0<\xi<1$, and $\mu>0$ (the scaling parameter in Hypothesis 3), which we collect in the triple

$$\Delta = (\mu, \rho, \xi). \tag{V.14}$$



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on $\mathcal{H}_{\mathrm{red}}$, where $W = \sum_{M+N \geqslant 1} W_{M,N}$, with

$$W_{M,N} = \int dk^{(M)}\, d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\, w_{M,N}[H_f; k^{(M)}, \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}). \tag{V.23}$$

Clearly, H in (V.22) uniquely determines an element $(E, T, \underline{W}) \in \mathcal{W}'_\Delta$, and we identify $H \equiv (E, T, \underline{W}) \in \mathcal{W}'_\Delta$ whenever this appears to be convenient. Furthermore, operators of the form (V.23) will be called (M, N)-*mononials* and the functions $w_{M,N}$ entering to their definition, the *coupling functions* of $W_{M,N}$. Since the correspondence between $W = \sum_{M+N \geqslant 1} W_{M,N}$ and $\underline{W} \in \bigoplus_{M+N \geqslant 1} \mathcal{W}_\Delta(M, N)$ is one-to-one, as well, we also write W instead of $\underline{W}$ whenever this appears to be convenient.

To control the z-dependence of the operators $H[z] \in \mathcal{W}'_\Delta$, we introduce the Banach space, $\mathcal{W}_\Delta$, of analytic families of bounded operators, $H\colon D_{1/2} \to \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$, parametrized by elements $H[z] \equiv (E[z], T[z], W[z]) \in \mathcal{W}'_\Delta$ with the property that

$$\|H[\cdot]\|_\Delta := \sup_{z \in D_{1/2}} \|(E[z], T[z; H_f], W[z])\|'_\Delta < \infty. \tag{V.24}$$

V.1.2. (*Unprojected*) *renormalization map on* $\mathcal{W}_\Delta$. In order to elucidate general features of the infrared renormalization problem studied in this chapter, we first introduce a formal renormalization map, $\hat{\mathcal{R}}_\rho$, defined on a subset of the Banach space $\mathcal{W}_\Delta$ and then sketch some properties of orbits under iterations of the map $\hat{\mathcal{R}}_\rho$ by identifying the fixed points of $\hat{\mathcal{R}}_\rho$ and the stable and instable manifolds through these fixed points. We define a cylinder $\hat{\mathcal{C}} \subseteq \mathcal{W}'_\Delta$ by

$$\begin{aligned}\hat{\mathcal{C}} := \{H[z] \in \mathcal{W}'_\Delta \mid |\arg E[z]| < \theta_0, |\partial_r T[z] - \lambda| \leqslant \delta, |\arg \lambda| < \theta_0,\\ |\lambda| > 0, \|W[z]\|'_\Delta \leqslant \varepsilon, |\arg z| \geqslant 4\theta_0\},\end{aligned} \tag{V.25}$$

where δ and ε are small constants (depending on λ), and $\theta_0 > 0$ is sufficiently large.

The map $\hat{\mathcal{R}}_\rho$ is defined by

$$\hat{\mathcal{R}}_\rho(H)[z] := \rho^{-1} U_\rho [f\chi_{H_f} < \rho (H[z] - z) + z\chi_{H_f < \rho_0}]\, U_\rho^*, \tag{V.26}$$

for $H[z] \in \hat{\mathcal{C}}$. The renormalization map $\hat{\mathcal{R}}_\rho$ has the following properties:

(1) The fixed points of $\hat{\mathcal{R}}_\rho$ are the operators in

$$\mathcal{F}\mathcal{P} := \{\lambda H_f \mid |\arg \lambda| < \theta_0\}. \tag{V.27}$$

(2) The tangent space at a point $\lambda H_f \in \mathscr{F}\mathscr{P}$ ($\lambda \neq 0$, $|\arg \lambda| < \theta_0$) can be split into a direct sum, $\mathscr{R} \oplus \mathscr{M} \oplus \mathscr{I}$, of a one-dimensional subspace, $\mathscr{R}$, of relevant perturbations, defined by

$$\mathscr{R} := \{ E[z] \cdot \mathbf{1} \,|\, |\arg E[z]| < \theta_0 \}, \tag{V.28}$$

a one-dimensional subspace of marginal perturbations,

$$\mathscr{M} := \{ \mu H_f \,|\, \mu \in \mathbb{C} \}, \tag{V.29}$$

and a co-dimension-2 subspace, $\mathscr{I}$, of irrelevant perturbations, defined by

$$\mathscr{I} := \{ \underline{W} \,|\, \| \underline{W} \|_{\Delta} < \infty \}. \tag{V.30}$$

(3) The expansion rate of $\hat{\mathscr{R}}_\rho$ in the direction of $\mathscr{R}$ is given by ρ^{-1}, in the direction of $\mathscr{M}$ it is $=0$, and the contraction rate of $\hat{\mathscr{R}}_\rho$ in the direction of $\mathscr{I}$ is $\rho^{\mu/2}$. An orbit of an operator in $\hat{\mathscr{C}}$ is sketched in Fig. V.2.

The interest in the renormalization map $\hat{\mathscr{R}}_\rho$ lies in the circumstance that it is *isospectral* in the sense of Theorem IV.1. Thus, in order to study, e.g., the resolvent set of a family $H[z] \in \hat{\mathscr{C}}$, we may study the resolvent set of $\hat{\mathscr{R}}^n_\rho(H)[z]$. Since the perturbation $W = \sum_{M+N \geqslant 1} W_{M,N}$ becomes small under iterations of $\hat{\mathscr{R}}_\rho$, the operators $\hat{\mathscr{R}}^n_\rho(H)[z]$ are simpler to analyze than the original operator $H[z]$.

A difficulty in analyzing orbits of families of operators in $\hat{\mathscr{C}}$ under iterations of $\hat{\mathscr{R}}_\rho$ is the divergence of such orbits in the direction of the relevant perturbations, $\mathscr{R}$. This difficulty can be avoided by projecting orbits along $\mathscr{R}$ onto the stable manifold of $\hat{\mathscr{R}}_\rho$ and by successive fine-tunig of the initial

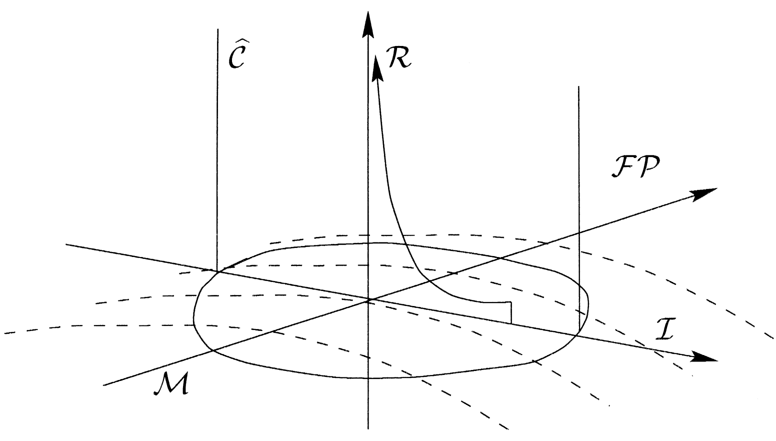

FIG. V.2. Orbit under $\mathscr{R}$ starting at H_0

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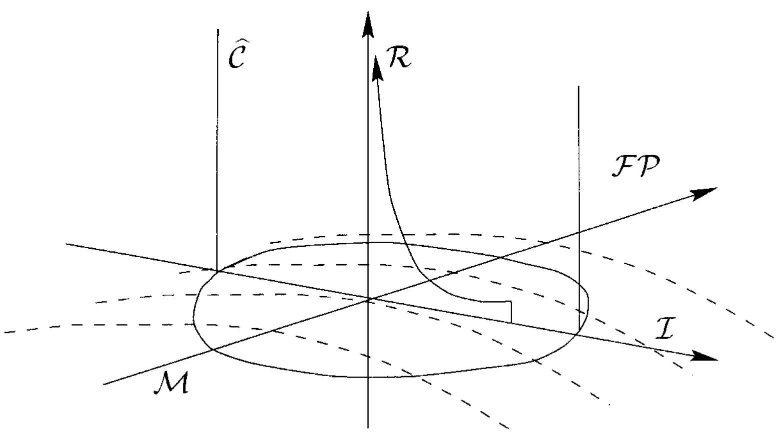

FIG. V.2. Orbit under $\mathscr{R}$ starting at H_0

value of the spectral parameter, z. Some details of our construction of such a modified renormalization map, $\mathscr{R}_\rho$, are described in the next subsection.

V.1.3. *Projected renormalization map on* $\mathscr{W}_\Delta$. We define a polydisc, $\mathscr{B}(\delta, \varepsilon)$, of operators in $\mathscr{W}_\Delta$ by

$$\mathscr{B}(\delta, \varepsilon) := \{(E, T, W) \in \mathscr{W}_\Delta \mid |\partial_r T - 1| \leqslant \delta, \; \|W\|_\Delta + |E| \leqslant \varepsilon\}. \tag{V.31}$$

Next, we pick $H \in \mathscr{B}(\frac{1}{8}, \rho/8)$ and define

$$Z: \mathscr{U}^{(\mathrm{in})} \to D_{1/2}, \qquad \zeta \mapsto \frac{1}{\rho}(\zeta - E[\zeta]), \tag{V.32}$$

where

$$\mathscr{U}^{(\mathrm{in})} := \{\zeta \in D_{1/2} \mid |\zeta - E[\zeta]| \leqslant \rho/2\} \tag{V.33}$$

(see Fig. V.3). We observe that $\zeta \in \mathscr{U}^{(\mathrm{in})}$, $H \in \mathscr{B}(\frac{1}{8}, (\rho/8))$, and $0 < \rho \leqslant 1/16$ imply that $|\zeta| \leqslant |E[\zeta]| + |\zeta - E[\zeta]| \leqslant 1/4$. Thus,

$$\mathscr{U}^{(\mathrm{in})} \subseteq D_{1/4}. \tag{V.34}$$

Then, Cauchy's estimate with contour on $\partial D_{1/2}$ yields that

$$|\partial_\zeta Z(\zeta) - 1| \leqslant 4 \sup_{\zeta \in D_{1/2}} \{|E[\zeta]|\} < \tfrac{1}{2}. \tag{V.35}$$

This proves that $Z: \mathscr{U}^{(\mathrm{in})} \to D_{1/2}$ is a bijection.

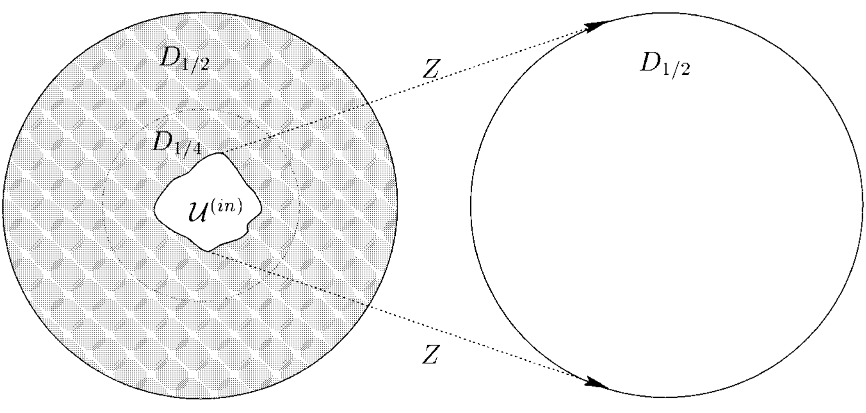

FIG. V.3. Rescaling of the spectral parameter.

with $Z_{(0)}(\zeta) = e^{i\vartheta} \rho_0^{-1}(\zeta - E_j)$ on $D(E_j, \rho_0/2)$, and, for $n \geqslant 1$,

$$Z_{(n)}: \mathscr{U}^{(\mathrm{in})}_{(n)} \to D_{1/2}, \qquad \zeta \mapsto \frac{1}{\rho} (\zeta - E_{(n-1)}[\zeta]), \tag{V.45}$$

where

$$\mathscr{U}^{(\mathrm{in})}_{(n)} := \{\zeta \in D_{1/2} \,|\, |\zeta - E_{(n-1)}[\zeta]| \leqslant \rho/2\}. \tag{V.46}$$

V.2. *The Family of Initial Operators*, $H_{(0)}[z]$

In this section, we investigate the family of operators $H_{(0)}[z]$ defined in (V.7)–(V.8). We show that this operator family $H_{(0)}[z]$ belongs to $\mathscr{B}(\frac{1}{16}, \rho/16)$, i.e., that (V.40) holds true.

V.2.1. *Bounds on the interaction.* The purpose of this subsection, is to estimate the operator norms of (M, N)-monomials $W_{M,N}$ (see Eq. (V.23)), with coupling functions $w_{M,N}$ in the Banach space $\mathscr{W}_\Delta(M, N)$.

Lemma V.1. *Let* $M + N \geqslant 1$, *and let* $W_{M,N}$ *be an* (M, N)*-monomial with coupling function* $w_{M,N} \in \mathscr{W}_\Delta(M, N)$ *(see Eq.* (V.23)*). Then, for all* $0 < \rho$, $\tilde{\rho} < 1$ *and* $\mu > -1$,

$$\begin{aligned} &\|(H_f + \rho)^{-1/2} \chi_{H_f < 1} W_{M,N} \chi_{H_f < 1} (H_f + \tilde{\rho})^{-1/2}\| \\ &\qquad \leqslant \frac{2\alpha \rho^{-(1/2)\,\delta_{M0}} \tilde{\rho}^{-(1/2)\,\delta_{N0}} C_\mu^{M+N} \|w_{M,N}\|_\Delta^{(\infty)}}{\Gamma[(\mu+1)\,M+1]^{1/2}\, \Gamma[(\mu+1)\,N+1]^{1/2}}, \end{aligned} \tag{V.47}$$

where $\Gamma(x+1) = x\Gamma(x)$ *is the Gamma function, and* $C_\mu := \sqrt{4\pi\Gamma[1+\mu]/3}$.

Proof. We pick $\phi = \chi_{H_f<1}\phi$, $\psi = \chi_{H_f<1}\psi \in \chi_{H_f<1}\mathscr{F}$ and consider

$$\begin{aligned} A^2(\phi, \psi) &:= |\langle \phi \,|\, \chi_{H_f<1} W_{M,N} \chi_{H_f<1} \psi \rangle|^2 \\ &= \left| \int dk^{(M)}\, d\tilde{k}^{(N)} \langle a(k^{(M)})\phi \,|\, w_{M,N}[H_f; k^{(M)}; \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)})\psi \rangle \right|^2. \end{aligned} \tag{V.48}$$

Remembering $a(k^{(M)}) \equiv \prod_{j=1}^M a(k_j)$ and $\omega(k^{(M)}) \equiv \sum_{j=1}^M \omega(k_j)$, the Pull-Through formulae (IV.63)–(IV.64) imply that

$$\begin{aligned} a(k^{(M)})\chi_{H_f<1} &= \chi[H_f + \omega(k^{(M)}) < 1]\, a(k^{(M)})\chi_{H_f<1} \\ &= \chi[\omega(k^{(M)}) < 1]\, a(k^{(M)})\chi_{H_f<1}, \end{aligned} \tag{V.49}$$

which, together with Schwarz' inequality and $\chi_{r<1} =: \chi_1[r]$, yields

$$A^2(\phi, \psi) \leqslant B^{(M)}(\phi) \cdot B^{(N)}(\psi) \cdot \int \chi_1[\omega(k^{(M)})]\, \chi_1[\omega(\tilde{k}^{(N)})] \times \{ \sup_{0<r<1} |w_{M,N}[r; k^{(M)}; \tilde{k}^{(N)}]|^2 \} \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)}, \tag{V.50}$$

where $B^{(0)}(\phi) := \|\phi\|^2$ and

$$B^{(M)}(\phi) := \int \|a(k^{(M)})\, \chi_1 \phi\|^2 \prod_{j=1}^{M} \omega(k_j)\, dk_j. \tag{V.51}$$

Another application of the Pull-Through formulae (IV.63)–(IV.64) then gives

$$\begin{aligned} B^{(M)}(\phi) &= \int \left\langle \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \,\middle|\, H_f \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \right\rangle \prod_{j=1}^{M-1} \omega(k_j)\, dk_j \\ &\leqslant \int \left\langle \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \,\middle|\, (H_f + \omega(k^{(M-1)})) \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \right\rangle \prod_{j=1}^{M-1} \omega(k_j)\, dk_j \\ &= \int \|a(k^{(M-1)})\, \chi_1 H_f^{1/2} \phi\|^2 \prod_{j=1}^{M-1} \omega(k_j)\, dk_j = B^{(M-1)}(H_f^{1/2} \phi) \\ &\leqslant B^{(M-2)}(H_f \phi) \leqslant \cdots \leqslant B^{(0)}(H_f^{M/2} \chi_{H_f<1} \phi) = \|H_f^{M/2} \chi_{H_f<1} \phi\|^2, \end{aligned} \tag{V.52}$$

denoting $\chi_1 \phi := \chi_{H_f<1} \phi$. Inserting the assumption on $w_{M,N}$, we estimate the integral on the right side of (V.50) by

$$\begin{aligned} &\int \chi_1[\omega(k^{(M)})]\, \chi_1[\omega(\tilde{k}^{(N)})] \{ \sup_{0<r<1} |w_{M,N}[r; k^{(M)}; \tilde{k}^{(N)}]|^2 \} \\ &\qquad \times \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)} \\ &\quad \leqslant (\|w_{M,N}\|_{\Delta}^{(\infty)})^2 \left(\int \chi_1[\omega(k^{(M)})] \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^{2-\mu}} \right) \\ &\qquad \times \left(\int \chi_1[\omega(\tilde{k}^{(N)})] \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^{2-\mu}} \right) \\ &\quad \leqslant (\|w_{M,N}\|_{\Delta}^{(\infty)})^2 \left(\frac{4\pi}{3} \right)^{M+N} \cdot \left(\int \chi[\omega_1 + \cdots + \omega_M < 1] \prod_{j=1}^{M} \frac{d\omega_j}{\omega_j^{-\mu}} \right) \\ &\qquad \cdot \left(\int \chi[\omega_1 + \cdots + \omega_N < 1] \prod_{j=1}^{N} \frac{d\omega_j}{\omega_j^{-\mu}} \right) \end{aligned} \tag{V.53}$$

$$= \frac{(\|w_{M,N}\|_{\Delta}^{(\infty)})^2\, ((4\pi/3)\, \Gamma[\mu+1])^{M+N}}{\Gamma[(\mu+1)\, M+1]\, \Gamma[(\mu+1)\, N+1]}. \tag{V.54}$$

Here, $\Gamma[x]$ denotes the Gamma function for $x>0$ and the last equality is derived in [5]. From inserting (V.53) and (V.52) into (V.50), we obtain

$$\|(H_f+\rho)^{-M/2}\chi_{H_f<1}W_{M,N}\chi_{H_f<1}(H_f+\rho)^{-N/2}\| \leqslant \frac{(\|w_{M,N}\|_{\Delta}^{(\infty)})^2\,((4\pi/3)\,\Gamma[\mu+1])^{(M+N)/2}}{\Gamma[(\mu+1)\,M+1]^{1/2}\,\Gamma[(\mu+1)\,N+1]^{1/2}}.$$

From here, the assertion follows trivially from

$$\left\|\frac{[H_f+\rho]^{1/2}}{[H_f+\rho]^{M/2}}\chi_{H_f<1}\right\| = \sup_{0\leqslant r<1}\{(r+\rho)^{(1-M)/2}\}\leqslant\sqrt{2}\,\rho^{-\delta_{M0}/2}. \quad \blacksquare \tag{V.55}$$

V.2.2. *The initial Hamiltonian $H_{(0)}$.* Now we proceed to showing that the operator family $H_{(0)}[z]$ belongs to $\mathscr{B}(\frac{1}{16},\rho/16)$, i.e., that (V.40) holds true. The main result of this section is

THEOREM V.2. *Assume Hypotheses 2 and 3 hold. Then, for any $0<\varepsilon_{(0)}\leqslant 1/16$, $0<\rho<1$, and any ρ_0, there is $g_0>0$ such that, for all $|g|\leqslant g_0$,*

$$H_{(0)}\in\mathscr{B}(\varepsilon_{(0)},\rho\varepsilon_{(0)}). \tag{V.56}$$

Proof. In what follows, we fix $\theta=i\vartheta$ with $\vartheta>0$ and do not display the dependence on this parameter. We also generally suppress the parameter z from the formulae. On the other hand, we explicitly display the functional dependence on the operator H_f as in

$$\bar{R}_0[H_f]=(H_{el}\otimes\mathbf{1}_f+e^{-i\vartheta}\mathbf{1}_{el}\otimes H_f-z)^{-1}P_0[H_f], \tag{V.57}$$

where (compare to (V.2))

$$P_0[H_f]\equiv P_0=P_{el,j}\otimes\chi_{H_f<\rho_0}, \tag{V.58}$$

and $P_{el,j}=|\psi_j\rangle\langle\psi_j|$ is the projection onto the eigenspace of H_{el} corresponding to E_j. We project an operator A on $\mathscr{H}_{el}\otimes\mathscr{F}$ onto an operator, $\langle A\rangle_{el,j}:=\langle\psi_j,A\psi_j\rangle_{\mathscr{H}_{el}}$ on $\mathscr{F}$ by means of $P_{el,j}\otimes\mathbf{1}_f$ as

$$P_{el,j}\otimes\langle A\rangle_{el,j}=(P_{el,j}\otimes\mathbf{1}_f)\,A(P_{el,j}\otimes\mathbf{1}_f). \tag{V.59}$$

In what follows we will also omit the trivial factor $P_{el,j}$ from the formulae.

First, for $|E_j-z|\leqslant\frac{1}{2}\rho_0$ and $\theta=i\vartheta$, $\vartheta>0$, we introduce the intermediate Hamiltonian $\tilde{H}_{\rm eff}[z]$ by

$$P_{el,j}\otimes\tilde{H}_{\rm eff}[z]:=e^{-i\vartheta}[f_{P_0}(H_g(\theta)-z)-E_j+z]\,P_0, \tag{V.60}$$

as in Eq. (V.1). Our task is to put this operator-family into the generalized normal form (V.9)–(V.10). To this end, we write

$$\widetilde{H}_{\mathrm{eff}}[z] =: \chi_{\rho_0}(\widetilde{E}_{\mathrm{eff}}[z] + \widetilde{T}_{\mathrm{eff}}[z; H_f] + \widetilde{W}_{\mathrm{eff}}[z])\chi_{\rho_0}, \tag{V.61}$$

where $\chi_s := \chi_{H_f < s}$, $\widetilde{E}[z] \in \mathbb{C}$, and $T[z; H_f]$ is a spectral function of H_f. They are given by

$$\widetilde{E}_{\mathrm{eff}}[z] := \tilde{w}^{\mathrm{eff}}_{0,0}[z; 0], \tag{V.62}$$

$$\widetilde{T}_{\mathrm{eff}}[H_f; z] := H_f + \tilde{w}^{\mathrm{eff}}_{0,0}[z; H_f] - \tilde{w}^{\mathrm{eff}}_{0,0}[z; 0], \tag{V.63}$$

$$\tilde{w}^{\mathrm{eff}}_{0,0}[s; z] := e^{i\vartheta}\langle f_{\chi_\rho}(H_g + \mathrm{e}^{-i\vartheta} s - z) - E_j + z\rangle_\Omega, \tag{V.64}$$

where $\langle \cdot \rangle_\Omega$ denotes the expectation value in the Fock vacuum state, Ω. Furthermore,

$$\widetilde{W}_{\mathrm{eff}}[z] := \sum_{M+N \geqslant 1} \widetilde{W}^{\mathrm{eff}}_{M,N}[z], \tag{V.65}$$

is a sum of (M, N)-monomials of the form

$$\widetilde{W}^{\mathrm{eff}}_{M,N}[z] := \int dk^{(M)}\, d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\, \tilde{w}^{\mathrm{eff}}_{M,N}[z; H_f; k^{(M)}, \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}). \tag{V.66}$$

Note that we have not rescaled the photon momenta by U_{ρ_0}, yet, and thus the coupling functions $\tilde{w}^{\mathrm{eff}}_{M,N}$ are maps

$$\tilde{w}^{\mathrm{eff}}_{M,N}\colon D(E_j, \rho_0/2) \times [0, \rho_0] \times B^{3M}_{\rho_0} \times B^{3N}_{\rho_0} \to \mathbb{C}. \tag{V.67}$$

Next, we determine the exact form of $\tilde{w}^{\mathrm{eff}}_{M,N}$ by a Wick-ordering procedure.

LEMMA V.3. *For $g\rho_0^{1/2} > 0$ sufficiently small, the coefficients, $\tilde{w}_{M,N}$, defined through* (V.61)–(V.66) *are given by*

$$\tilde{w}_{M,N}[r; z; k^{(M)}; \tilde{k}^{(N)}]$$
$$= \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{\substack{m_l + p_l + n_l + q_l = 1, 2;\\ l = 1, \ldots, L}} \delta_{\sum_{l=1}^L m_l, M}\, \delta_{\sum_{l=1}^L n_l, N}$$
$$\times \prod_{l=1}^{L} \left\{ \binom{m_l + p_l}{p_l} \binom{n_l + q_l}{q_l} \right\} \{\widetilde{D}_L[r; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}]\}^{symm}_{M,N}, \tag{V.68}$$

where

$$\widetilde{D}_L[r; \{W^{m_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^L]$$

$$:= e^{i\vartheta} \prod_{l=1}^{L} (-\lambda)^{m_l+p_l+n_l+q_l} \cdot \langle \varphi_{el} \otimes \Omega \,|\, W^{m_1, n_1}_{p_1, q_1}[k_1^{(m_1)}; \tilde{k}_1^{(n_1)}]\, \bar{R}_0[H_f+\mu_1]$$

$$\cdots \bar{R}_0[H_f+\mu_{L-1}]\, W^{m_L, n_L}_{p_L, q_L}[k_L^{(m_L)}; \tilde{k}_L^{(n_L)}]\, \varphi_{el} \otimes \Omega \rangle, \tag{V.69}$$

$$W^{m_l, n_l}_{p_l, q_l}[k_l^{(m_l)}; \tilde{k}_l^{(n_l)}] := \int dx_l^{(p_l)}\, d\tilde{x}_l^{(q_l)}\, G_{m_l+p_l, n_l+q_l}[k_l^{(m_l)}, x_l^{(p_l)}; \tilde{k}_l^{(n_l)}, \tilde{x}_l^{(q_l)}]$$

$$\otimes a^\dagger(x_l^{(p_l)})\, a(\tilde{x}_l^{(q_l)}), \tag{V.70}$$

$$\mu_l := r + \sum_{j=1}^{l} \omega(\tilde{k}_j^{(n_j)}) + \sum_{j=l+1}^{L} \omega(k_j^{(m_j)}), \tag{V.71}$$

$\bar{R}_0$ *is defined in* (V.57), *and*

$$\{A[k^{(M)}; \tilde{k}^{(N)}]\}^{symm}_{M, N}$$

$$:= \frac{1}{M!\, N!} \sum_{\pi \in \mathcal{S}_m} \sum_{\sigma \in \mathcal{S}_N} A[k_{\pi(1-)}, ..., k_{\pi(M)}, \tilde{k}_{\sigma(1)}, ..., \tilde{k}_{\sigma(N)}]. \tag{V.72}$$

Proof. We use definition (IV.4) of the Feshbach map f_{P_0} and expand the resolvent $(\bar{P}_{\rho_0} H_g(\theta)\, \bar{P}_{\rho_0} - z)^{-1} = (H_0 \bar{P}_{\rho_0} + \bar{P}_{\rho_0} W_g \bar{P}_{\rho_0} - z)^{-1}$, entering $f_{P_{\rho_0}}(H_g(\theta) - z)$, in a Neumann series in the operator $\bar{P}_{\rho_0} W_g \bar{P}_{\rho_0}$:

$$\widetilde{H}[z] = H_f - \sum_{L=1}^{\infty} (-e^{-i\vartheta})^L \chi_{H_f < \rho_0} \langle W_g(\bar{R}_0[H_f]\, W_g)^{L-1} \rangle_{el, j}\, \chi_{H_f \leqslant \rho_0} \tag{V.73}$$

where we used the notation introduced in (V.57)–(V.59). Since, for $|z - E_0| \leqslant \rho_0/2$,

$$|H_0 - z| \geqslant |H_0 - E_j| - \frac{\rho_0}{2} \geqslant \min\left\{\frac{1}{2}, \frac{\delta}{\rho_0} \sin \vartheta\right\} (H_f + \rho_0), \tag{V.74}$$

on $\chi_{H_f < \rho_0} \mathscr{F}$, where δ is the distance from E_j to the rest of the spectrum of H_{el}, Estimate (V.47) implies that the sum on the r.h.s. of (V.73) converges in norm, provided $|g|\, \rho_0^{-1/2}$ is sufficiently small. Next we use the identity

$$W_{M_1, N_1} \bar{R}_0[H_f]\, W_{M_2, N_2} \bar{R}_0[H_f] \cdots \bar{R}_0[H_f]\, W_{M_L, N_L}$$

$$= \sum_{m_1=0}^{M_1} \sum_{n_0=0}^{N_1} \cdots \sum_{m_L=0}^{M_L} \sum_{n_L=0}^{N_L} \int \prod_{l=1}^{L} \left\{ dk_l^{(M_l - m_l)}\, d\tilde{k}_l^{(N_l - n_l)} \binom{M_l}{m_l} \binom{N_l}{n_l} \right\} \tag{V.75}$$

which follows from the Pull-Through formula

$$f[H_f]\, a^\dagger(k) = a^\dagger(k)\, f[H_f + \omega(k)], \tag{V.76}$$

$$a(k)\, f[H_f] = f[H_f + \omega(k)]\, a(k), \tag{V.77}$$

and the following standard Wick's Theorem

$$\prod_{j=1}^{N} a^{\sigma_j}(k_j) = \sum_{Q \subseteq \{1, ..., N\}} \left\langle \prod_{j \in \{1, ..., N\} \setminus Q} a^{\sigma_j}(k_j) \right\rangle_{\Omega} :\prod_{j \in Q} a^{\sigma_j}(k_j):, \tag{V.78}$$

where $a^+ := a^\dagger$, $a^- := a$, and

$$:\prod_{j=1}^{N} a^{\sigma_j}(k_j): := \prod_{\substack{j=1, \\ \sigma_j = +}}^{N} a^\dagger(k_j) \prod_{\substack{j=1, \\ \sigma_j = -}}^{N} a(k_j) \tag{V.79}$$

In fact, Eqs. (V.76)–(V.79) imply a generalization of Wick's theorem, for operators of the form (V.23), which can be used to derive (V.75) (see [5] for details). From (V.75) we can directly read off the coefficients $\tilde{w}_{M,N}$. ∎

LEMMA V.4. *Assume that Hypotheses* 2 *and* 3 *hold, i.e., the function* J *obeys the bound*

$$J(k) \leqslant |k|^{(\mu-1)/2}. \tag{V.80}$$

Then, for sufficiently small $g\rho_0^{-1/2} > 0$ *and* $M + N \geqslant 0$,

$$|\tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]| \leqslant \left(\frac{Cg}{\rho_0^{1/2}}\right)^{M+N+2\delta_{M+N,0}} \rho_0^{1-(1/2)(M+N)} \cdot \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}. \tag{V.81}$$

and

$$\int |\partial_r \tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]| \prod_{i=1}^{M} \frac{dk_i}{|k_i|^{(3+\mu)/2}} \prod_{i=1}^{N} \frac{d\tilde{k}_i}{|\tilde{k}_i|^{(3+\mu)/2}} \leqslant \left(\frac{Cg}{\rho_0^{1/2}}\right)^{M+N+2\delta_{M+N,0}} \rho_0^{1-(1/2)(M+N)}. \tag{V.82}$$

Proof. We only give the proof of (V.81). The proof of (V.82) is given in [5]. To show (V.81), we need the following slight generalization of (IV.44) in Lemma IV.4:

$$\|K^{-1/2}\, W^{m,n}_{m+p,n+q}(k^{(m)}; \tilde{k}^{(n)})\, K^{-1/2}\|$$
$$\leqslant C_0 g^{m+n+p+q} \rho_0^{-(1/2)(1+\delta_{p+q,0})} \cdot \prod_{j=1}^{m} J(k_j) \cdot \prod_{j=1}^{n} J(\tilde{k}_j), \tag{V.83}$$

where $K = |H_{el} - E_j| \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + \rho_0$ is defined in (IV.36) and

$$W^{m,n}_{m+p,n+q}(k^{(m)}; \tilde{k}^{(n)})$$
$$:= g^{m+n+p+q} \int dx^{(p)}\, dx^{(q)}\, G^{(\theta)}_{m+p,n+q}(k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}) \otimes a^\dagger(x^{(p)})\, a(\tilde{x}^{(q)}), \tag{V.84}$$

for $m+p+n+q=1$ or $=2$ (see (IV.17) and (IV.18)). The proof goes along the same lines as the one of Lemma IV.4.

Using the fact that $\psi_j \otimes \Omega = \rho_0^{1/2} K^{-1/2} \psi_j \otimes \Omega$, we estimate

$$|\tilde{D}_L[r; \{W^{m_l,n_l}_{m_l+p_l,n_l+q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}]|$$
$$= \rho_0 \,|\langle \varphi_j \otimes \Omega \,|\, (K^{-1/2} W_1 (K^{-1/2} + \mu_1)^{-1/2})(\bar{R}_0[H_f + \mu_1](K+\mu_1))$$
$$\cdots (\bar{R}_0[H_f + \mu_{L-1}](K + \mu_{L-1})) \cdot ((K+\mu_{L-1})^{-1/2}\, W_L(K^{-1/2}))\, \varphi_j \otimes \Omega\rangle$$
$$\leqslant \rho_0 \cdot \prod_{l=1}^{L} \|(K+\mu_{l-1})^{-1/2}\, W_l (K+\mu_l)^{-1/2}\| \cdot \prod_{l=1}^{L-1} \|\bar{R}_0[H_f + \mu_l](K+\mu_l)\|$$
$$\leqslant \rho_0 \prod_{l=1}^{L} \left(\frac{(Cg)^{m_l+p_l+n_l+q_l}}{\rho_0^{(1/2)+(1/2)\,\delta_{p_l+q_l,0}}} \right) \cdot \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}, \tag{V.85}$$

where we set $\mu_0 = \mu_L = 0$, and in the last inequality we make use of (V.83), (V.80) and (V.6). Now, we observe that $m_l + n_l + p_l + q_l \geqslant 1$ implies that $1 + \delta_{(p_l+q_l),0} \leqslant 2(m_l+n_l) + p_l + q_l$ and hence

$$\frac{g^{m_l+n_l+p_l+q_l}}{\rho_0^{(1/2)+(1/2)\,\delta_{p_l+q_l,0}}} \leqslant \left(\frac{g}{\rho_0^{1/2}} \right)^{m_l+n_l+p_l+q_l} \cdot \rho_0^{-(m_l+n_l)/2}. \tag{V.86}$$

Thus, for any $a > 0$ and g sufficiently small such that $2Ca^2 g \leqslant \rho_0^{1/2}$,

$$|\tilde{D}_L[r; \{W^{m_l,n_l}_{m_l+p_l,n_l+q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}]|$$
$$\leqslant -\left(\frac{2Ca^2 g}{\rho_0^{1/2}} \right)^{\max\{M+N,\, L\}} \rho_0^{1-(M+N)/2}\, a^{-L} (2a)^{-m_l-n_l-p_l-q_l}$$
$$\times \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}. \tag{V.87}$$

We insert this estimate into (V.68) and observe that $\max\{M+N, L\} \geqslant M+N+2\delta_{(M+N),0}$, since $L\geqslant 2$, for $M+N=0$. Additionally estimating the binomial coefficients in (V.68) by $2^{m_l+p_l+n_l+q_l}$, we can sum up the series

$$|\tilde{w}_{M,N}[r;z;k^{(M)};\tilde{k}^{(N)}]| \left(\frac{2Ca^2g}{\rho_0^{1/2}}\right)^{-M-N-2\delta_{(M+N),0}}$$

$$\times \rho_0^{(1/2)(M+N)-1} \prod_{j=1}^{M} |k_j|^{(1-\mu)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(1-\mu)/2}$$

$$\leqslant \sum_{L=1}^{\infty} a^{-L} \sum_{\substack{m_l+p_l+n_l+q_l=1,2;\\ l=1,\dots,L}} \prod_{l=1}^{L} \left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\right\} (2a)^{-m_l-n_l-p_l-q_l}$$

$$\leqslant \sum_{L=1}^{\infty} \left(\frac{a^4}{(a-1)^5}\right)^{-L}, \tag{V.88}$$

arriving at (V.81), upon the choice $a:=5$. ∎

Now, we return to the Hamiltonian $H_{(0)}[z]$. Using its definition (V.8), we find that

$$H_{(0)}[z]-z=\frac{1}{\rho_0} U_{\rho_0} f_{P_0}(\tilde{H}_{\mathrm{eff}}[Z_{(0)}^{-1}(z)]-Z_{(0)}^{-1}(z))\, \chi_{H_f<\rho_0} U_{\rho_0}^{*}, \tag{V.89}$$

where, recall,

$$Z_{(0)}: D(E_j, \rho_0/2) \to D_{1/2}, \qquad \zeta \mapsto \frac{e^{i\vartheta}}{\rho_0}(\zeta-E_j). \tag{V.90}$$

This relation implies that $H_{(0)}[z]$ is of the form (V.9)–(V.10) with coupling functions $w_{M,N}^{(0)}$ given by

$$w_{M,N}^{(0)}[z;r;k^{(M)};\tilde{k}^{(N)}]=\rho_0^{(3/2)(M+N)-1}\, \tilde{w}_{M,N}^{\mathrm{eff}}[Z_{(0)}^{-1}(z);\rho_0 r;\rho_0 k^{(M)};\rho_0\tilde{k}^{(N)}]. \tag{V.91}$$

By insertion of (V.81)–(V.82) into (V.91) and taking the definitions (V.19)–(V.20) into account, we obtain the following corollary of Lemma V.4.

COROLLARY V.5. *We require Hypothesis* 2 *and* 3. *Then, for sufficiently small* $g\rho_0^{-1/2}>0$ *and* $M+N\geqslant 0$,

$$\|w_{M,N}^{(0)}[z]\|_{\Delta}^{(\infty)} \leqslant \left(\frac{Cg}{\rho_0^{1/2}}\right)^{(M+N+1)/2} \cdot (Cg\rho_0^{\mu})^{M+N}, \tag{V.92}$$

and, for $M+N \geqslant 1$,

$$\|\partial_r w^{(0)}_{M,N}[z]\|^{(1)}_{\Delta} \leqslant \left(\frac{Cg}{\rho_0^{1/2}}\right)^{(M+N+1)/2} \cdot (Cg\rho_0^{\mu})^{M+N}. \tag{V.93}$$

Taking into account that ρ_0 is bounded above by the distance of E_j to $\sigma(H_{el})\backslash\{E_j\}$, we observe that a sufficiently small choice of $g\rho_0^{-1/2}$, together with (V.92)–(V.93), implies (V.56). This completes the proof of Theorem V.2. ∎

We point out that our analysis in Chapter IV, in particular (IV.96), gives us control over all $z \in D(E_j, \rho_0/2)$, provided $\rho_0 \gg g^2$.

Corollary V.5 and Lemma V.1 imply that, for $M+N \geqslant 1$ and $0<\rho<1$,

$$\begin{aligned}\|(H_f+\rho)^{-1/2}\,&\chi_{H_f<1}\,W^{(0)}_{M,N}\chi_{H_f<1}(H_f+\rho)^{-1/2}\|\\ &\leqslant \left(\frac{Cg}{\rho_0^{1/2}}\right)^{(M+N+1)/2}\left(\frac{\rho_0^{1+\mu}}{\rho}\right)^{(M+N)/2}.\end{aligned} \tag{V.94}$$

Here, to pass from (V.47) and (V.92) to (V.94), we additionally used that $M+N \geqslant 1$ implies $\delta_{M0}+\delta_{N0} \leqslant 1$ and that $\Gamma[\mu+1]^m \leqslant \Gamma[(\mu+1)\,m+1]$. This last estimate appears to be rather rough, as we do not make use of the superexponential growth of the Gamma function. Indeed, in the renormalization scheme we present in the next section, this superexponential growth becomes important, but for our present consideration the weaker estimate (V.94) is sufficient. Since $W_{(0)} = \sum_{M+N\geqslant 1} W^{(0)}_{M,N}$, we obtain from (V.94)

LEMMA V.6. *Assume Hypotheses* 2 *and* 3, *suppose that* $g\rho_0^{-1/2}$ *is sufficiently small and that* $0<\rho_0^{1+\mu} \leqslant \rho_1 < \rho_0 \leqslant 1$. *Then*

$$\|(H_f+\rho_1)^{-1/2}\,\chi_{H_f<1}\,W_{(0)}\chi_{H_f<1}(H_f+\rho_1)^{-1/2}\| \leqslant \frac{Cg\rho_0^{\mu/2}}{\rho_1^{1/2}}. \tag{V.95}$$

Note the fact that μ must be strictly positive to ensure that $\rho_1<\rho_0$. This makes the heuristic discussion given in the introduction precise.

V.3. *Properties of the Renormalization Map* $\mathscr{R}_\rho$

In this section, we study the renormalization map $\mathscr{R}_\rho$, defined in Eqs. (V.36)–(V.38).

V.3.1. *The domain of definition of* $\mathscr{R}_\rho$. The purpose of this subsection is to prove that $\mathscr{R}_\rho$ is defined on $\mathscr{B}(\frac{1}{8}, \rho/16)$.

LEMMA V.7. *The Feshbach Map* f_{χ_ρ} *and thus* $\mathscr{R}_\rho$ *is defined on* $\mathscr{B}(\frac{1}{16}, \rho/16)$.

Proof. Let $H \equiv (E, T, W) \in \mathscr{B}(\frac{1}{16}, \rho/16)$. Then, for $H_f \geqslant \rho_0$,

$$|T[z; H_f] + E[z] - z| \geqslant |T[z; H_f]| - \frac{\rho}{8} \geqslant \frac{3}{4}(H_f + \rho). \tag{V.96}$$

On the other hand, by Lemma V.1,

$$\|(H_f + \rho)^{-1/2} \chi_{H_f<1} W \chi_{H_f<1} (H_f + \rho)^{-1/2}\| \leqslant \frac{32(\pi/3)^{1/2} \xi\varepsilon}{\rho^{1/2}}. \tag{V.97}$$

Hence, if $\varepsilon\rho^{-1/2} < \frac{1}{4}(\pi/3)^{-1/2}$, then the operator $\bar{\chi}_\rho(H[z] - z)\bar{\chi}_\rho$ is invertible on $\mathscr{H}_{\text{red}} = \bar{\chi}_\rho \mathscr{F}$. Indeed, this follows by expanding $\bar{\chi}_\rho(H[z] - z)\bar{\chi}_\rho$ into a Neumann series in $\bar{\chi}_\rho W \bar{\chi}_\rho$. The invertibility of $\bar{\chi}_\rho(H[z] - z)\bar{\chi}_\rho$ and the definition (IV.4) of f_{χ_ρ} imply the statement of the lemma. ∎

V.3.2. *The contraction property of the renormalization map $\mathscr{R}_\rho$.* In this subsection we prove that $\mathscr{R}_\rho$ is a contraction on small balls $\mathscr{B}(\frac{1}{16}, \rho/16)$ around the fixed point $H_f \in \mathscr{F}\mathscr{P}$. As before, we identify an operator family $(E, T, \underline{W}) \in \mathscr{W}_\Delta$ with its generalized normal form $H = \chi_1(E + T + W)\chi_1$. Denote $H_0 := (E + T)\chi_1$. The desired contraction property will be derived from the following

THEOREM V.8. *Let $\mu > 0$ and pick ρ and ξ satisfying $0 < \rho^\mu \leqslant \frac{1}{16}$ and $0 < 2\sqrt{\pi}\,\xi \leqslant \min\{1/12, \rho^{(3+\mu)/4}\}$. Assume furthermore that $0 < \delta \leqslant 1/8$ and $\varepsilon\rho^{-1/2} \leqslant 1/12800$. Then, for $H \in \mathscr{B}(\delta, \varepsilon)$, we have*

$$\|\mathscr{R}_\rho(H) - \mathscr{R}_\rho(H_0)\|_\Delta \leqslant \eta\, \|H - H_0\|_\Delta, \tag{V.98}$$

where

$$\eta = 8\rho^{\mu/2}. \tag{V.99}$$

Proof. Here we only sketch some key ideas of the proof. The complete proof, including the precise determination of the parameters $\mu, \rho, \xi, \varepsilon, \delta$, and η can be found in [5]. Denote $\tilde{H} := \mathscr{D}_\rho(H)$ and $\hat{H} := \mathscr{R}_\rho(H)$, so that $\hat{H}[z] = \mathscr{S}_\rho(\tilde{H}[Z^{-1}(z)])$. Write $\tilde{H}$ in the generalized normal form: $\tilde{H} = \chi_\rho(\tilde{E} + \tilde{T} + \tilde{W})\chi_\rho$. Then the coupling functions, $\tilde{w}_{M,N}$, entering $\tilde{W}$ can be represented, for $(k^{(M)}, \tilde{k}^{(N)}) \in B_\rho^M \times B_\rho^N$, as

$$\tilde{w}_{M,N}[z; r; k^{(M)}, \tilde{k}^{(N)}] = \tilde{w}^T_{M,N}[z; r; k^{(M)}, \tilde{k}^{(N)}] + \Delta\tilde{w}_{M,N}[z; r; k^{(M)}, \tilde{k}^{(N)}], \tag{V.100}$$

where $\tilde{w}^T_{M,N} := w_{M,N}$ and $\Delta\tilde{w}_{M,N}$ is given by

$$\begin{aligned}
&\Delta\tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}] \\
&\quad = \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{m_l+p_l+n_l+q_l=1,2;\\ l=1,\dots,L}} \delta_{\sum_{l=1}^L m_l, M} \delta_{\sum_{l=1}^L n_l, N} \\
&\qquad \times \prod_{l=1}^{L} \left\{ \binom{m_l+q_l}{p_l} \binom{n_l+q_l}{q_l} \right\} \{ \tilde{D}_L[r; \{ W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}] \}^{symm}_{M,N},
\end{aligned} \tag{V.101}$$

in analogy to (V.69). We remark that the representation (V.101) is valid for $\Delta w_{0,0}$, as well. We obtain, for $M+N \geqslant 1$, that

$$|w_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]| \leqslant \varepsilon \xi^{M+N} \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}, \tag{V.102}$$

and we estimate $|\tilde{D}_L[\cdots]|$ in a similar way as in the proof of Lemma V.4, using that $\xi \leqslant 1/4$:

$$\begin{aligned}
&|\tilde{D}_L[r; \{ W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}; \{ \bar{R}_0 \}_{l=1}^{L-1}]| \\
&\quad \leqslant \rho \prod_{l=1}^{L} \frac{C\varepsilon \xi^{m_l+n_l+p_l+q_l}}{\rho^{(1/2)+(1/2)\,\delta_{p_l+q_l,0}}} \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2} \\
&\quad \leqslant \rho \cdot \left(\frac{4C\varepsilon}{\rho} \right)^{L} \cdot \left(\frac{4\xi}{\rho^{1/2}} \right)^{M+N} \cdot \prod_{l=1}^{L} \left(\frac{1}{4} \right)^{m_l+n_l+p_l+q_l} \\
&\qquad \cdot \prod_{j=1}^{m} |k_j|^{(\mu-)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}.
\end{aligned} \tag{V.103}$$

Moreover,

$$\begin{aligned}
&\sum_{\substack{m_l+\cdots+q_l \geqslant 1\\ l=1,\dots,L}} \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \left(\frac{1}{4} \right)^{m_1+n_l+p_l+q_l} \right\} \\
&\qquad \leqslant \left(\sum_{m=0}^{\infty} 2^{-m} \right)^{4L} = (16)^L.
\end{aligned} \tag{V.104}$$

Thus, putting together (V.103) and (V.104) and summing over $L \geqslant 2$ with $16C\varepsilon < \rho^{1/2}$, we obtain

$$\begin{aligned}
&|(\tilde{w}_{M,N} - w_{M,N})[z; r; k^{(M)}; \tilde{k}^{(N)}]| \\
&\qquad \leqslant C\varepsilon^2 \left(\frac{4\xi}{\rho^{1/2}} \right)^{M+N} \cdot \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}.
\end{aligned}$$

We define, in accordance with the equation $\hat{H}[z] = \mathcal{S}_\rho(\tilde{H}[Z^{-1}(z)])$,

$$\hat{w}^T_{M,N}[z; r; k^{(M)}, \tilde{k}^{(N)}] := \rho^{(3/2)(M+N)-1} \tilde{w}^T_{M,N}[Z^{-1}(z); \rho r; \rho k^{(M)}, \rho \tilde{k}^{(N)}], \tag{V.105}$$

$$\Delta\hat{w}^T_{M,N}[z; r; k^{(M)}, \tilde{k}^{(N)}] := \rho^{(3/2)(M+N)-1} \Delta\tilde{w}_{M,N}[Z^{-1}(z); \rho r; \rho k^{(M)}, \rho \tilde{k}^{(N)}], \tag{V.106}$$

so that $\hat{w}_{M,N} = \hat{w}^T_{M,N} + \Delta\hat{w}_{M,N}$, and we obtain from (V.102)

$$|\hat{w}^T_{M,N}[z; r; k^{(M)}, \tilde{k}^{(N)}]| \leqslant \frac{\varepsilon}{\rho} (\xi \rho^{1+(\mu/2)})^{M+N} \cdot \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}. \tag{V.107}$$

From (V.104), we get

$$|\Delta\hat{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]| \leqslant C\varepsilon^2 (4\xi \rho^{(1/2)+(\mu/2)})^{M+N} \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}. \tag{V.108}$$

Since $\rho^{\mu/2} < 1/16$ (actually, $\rho^{(1+\mu)/2} \leqslant 1/4$ would suffice here, but not for the estimate on $\partial_r \hat{w}_{M,N}$), by assumption, Estimate (V.108) yields, for $M+N \geqslant 1$,

$$|\hat{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]| \leqslant \rho^{\mu/2} \varepsilon [1 + C\varepsilon] \cdot \xi^{M+N} \prod_{j=1}^{M} |k_j|^{(\mu-1)/2} \prod_{j=1}^{N} |\tilde{k}_j|^{(\mu-1)/2}. \tag{V.109}$$

We observe that the requirement $\mu > 0$ is forced upon us by the behaviour of $\hat{w}^T_{0,1}$ and $\hat{w}^T_{1,0}$ under renormalization—all other terms renormalize to 0 with a higher power of ρ. Separately, we note that (V.108) with $M = N = 0$, the inequality $\varepsilon^2 \leqslant \rho^{1/2} \varepsilon$, and the relation $w^{(0)}_{0,0}[z; 0] := \rho^{-1} w_{0,0}[Z^{-1}(z); 0]$ yield, for $z \in D_{1/2}$

$$|\Delta\hat{w}_{0,0}[z; r]| \leqslant C\varepsilon^2 \leqslant C\rho^{1/2} \varepsilon. \tag{V.110}$$

Thus, for $z \in D_{1/2}$

$$|\hat{E}[z]|, |\hat{T}[z; r]| \leqslant C\rho^{1/2} \varepsilon, \tag{V.111}$$

which, by Cauchy's estimate, implies

$$|\partial_z \hat{E}[z]|, |\partial_z \hat{T}[z; r]| \leqslant 4C\rho^{1/2} \varepsilon < 1/6, \tag{V.112}$$

for all $z \in \mathcal{U}^{(\text{in})} \subseteq D_{1/4}$ and a sufficiently small choice of $\varepsilon \rho^{1/2}$.

Now we turn to the estimate of $|\partial_r \hat{T} - 1|$. Let us return to (V.96). At a first glance, it seems that all that is required to prove (V.96) is a bound on $|r^{-1}\hat{T}[z; r] - 1|$. Indeed, for $r \geqslant \rho$, we may estimate,

$$|\hat{T}[z; r] - \hat{E}[z] - z| \geqslant r(1 - \sup_{[0,1]} |r^{-1}\hat{T}[z; r] - 1|) - \frac{\rho}{2}$$

$$\geqslant r\left(\frac{1}{2} - \sup_{[0,1]} |r^{-1}\hat{T}[z; r] - 1|\right) > 0,$$

provided $|r^{-1}\hat{T}[z; r] - 1|$ is sufficiently small. However, a bound on $|r^{-1}\hat{T}[z; r] - 1|$ requires control of $|\partial_r T[z; r] - 1|$ and of $|\partial_r w_{M,N}[z, ...]|$ for arbitary values of r in the interval $[0, 1]$, as we show in an example below. Thus to ensure that $\mathscr{R}$ maps a space of Hamiltonians into itself, this space has to be equipped with a norm that yields control over the derivatives $|\partial_r T - 1|$ and $|\partial_r w_{M,N}|$. This is reflected in our choice of the norms $\|\cdot\|^{(\infty)}$ and $\|\cdot\|^{(1)}$, defined in (V.19) and (V.20). To illustrate the key idea of the proof we study one contribution to $\Delta\tilde{w}_{0,0}$ given by (surpressing the z-dependence in our notation)

$$S(r) := \langle W^{0,0}_{0,1} \bar{R}_0[H_f + r]\, W^{0,0}_{1,0} \rangle_\Omega$$

$$= \int w_{0,1}[r; x] \frac{\chi[r + \omega(x) \geqslant \rho]}{T[r + \omega(x)] - Z} w_{1,0}[r; x]\, dx. \qquad \text{(V.113)}$$

First, we observe that the derivative of S at $r = 0$ depends on the derivative of T at an arbitrary point, $\omega(x) \in [0, 1]$. Next, we compute the derivative of S:

$$\partial_r S(r) = \int w_{0,1}[r; x] \left(\frac{\delta(r + \omega(x) - \rho)}{T[\rho] + E - z}\right) w_{1,0}[r; x]\, dx$$

$$- \int w_{0,1}[r; x] \left(\frac{\partial_r T[r + \omega(x)]\, \chi[r + \omega(x) \geqslant \rho]}{(T[r + \omega(x)] + E - z)^2}\right) w_{1,0}[r; x]\, dx$$

$$+ \int \partial_r w_{0,1}[r; x] \left(\frac{\chi[r + \omega(x) \geqslant \rho]}{T[r + \omega(x)] + E - z}\right) w_{1,0}[r; x]\, dx$$

$$+ \int w_{0,1}[r; x] \left(\frac{\chi[r + \omega(x) \geqslant \rho]}{T[r + \omega(x)] + E - z}\right) \partial_r w_{1,0}[r; x]\, dx. \qquad \text{(V.114)}$$

From (V.114), we draw the following conclusions:

(a) The first term on the right side of (V.114) is given by

$$\int_{S^2=\{|n|=1\}} w_{0,1}[r;(\rho-r)n]\, w_{1,0}[r;(\rho-r)n]\, \frac{(\rho-r)^2\, d^2n}{T[\rho]+E-z} \tag{V.115}$$

and is bounded by $C\varepsilon^2\xi^2\rho^\mu$, for all $0\leqslant r\leqslant\rho$.

(b) The second term is bounded by $C\varepsilon^2\xi^2\rho^{-1}$.

(c) We bound the third term of (V.114) as follows:

$$\begin{aligned}
&\left|\int \partial_r w_{0,1}[r;x]\left(\frac{\chi[r+\omega(x)\geqslant\rho]}{T[r+\omega(x)]+E-z}\right) w_{1,0}[r;x]\,dx\right| \\
&\quad\leqslant \|w_{1,0}\|^{(\infty)}\cdot\rho^{-1}\cdot\int\frac{|\partial_r w_{0,1}[r;x]|\,dx}{|x|^{(\mu-1)/2}} \\
&\quad\leqslant \rho^{-1}\cdot\|w_{1,0}\|^{(\infty)}\cdot\|\partial_r w_{0,1}\|^{(1)}\leqslant C\varepsilon^2\xi\rho^{-1}(4\pi\xi\rho^{1/2}).
\end{aligned} \tag{V.116}$$

The fourth term is bounded similarly.

From summing the terms in (a)–(c) we obtain

$$|\partial_r S|\leqslant C\left(\frac{\varepsilon}{\rho^{1/2}}\right)^2\xi^2, \tag{V.117}$$

for all $0\leqslant r\leqslant\rho$. To bound a general contribution

$$\partial_r\langle W^{m_1,m_1}_{p_1,q_1}\bar{R}_0[H_f+\mu_1]\cdots\bar{R}_0[H_f+\mu_{L-1}]\, W^{m_L,n_L}_{p_L,q_L}\rangle_\Omega \tag{V.118}$$

to $\Delta\tilde{w}_{M,N}$, we evaluate the derivative using Leibniz' rule and estimate the resulting terms as in (a)–(c) above. Here, an additional problem arises from the large number of terms generated by writing each contribution to (V.118) of the form

$$\begin{aligned}
&\langle (W_1\bar{R}_0[H_f+\mu_1]\cdots\bar{R}_0[H_f+\mu_{l-2}]\, W_{l-1}\bar{R}_0[H_f+\mu_{l-1}]) \\
&\quad\times\partial_{H_f}(W_l\bar{R}_0[H_f+\mu_l])(W_{l+1}\bar{R}_0[H_f+\mu_{l+1}] \\
&\quad\cdots\bar{R}_0[H_f+\mu_{L-1}]\, W_L)\rangle_\Omega|_{r=H_f},
\end{aligned} \tag{V.119}$$

with $W_j := W^{m_j,n_j}_{p_j,q_j}$, as a sum of Wick-contractions between the factor to the left of $\partial_{H_f}(W_l\bar{R}_0[H_f+\mu_l])$ and the factor to its right. This problem is solved by using (V.47): If $n!$ terms are generated by n contracted creation and annihilation operators, then each term is bounded by $\Gamma[(\mu+1)\,n+1]^{-1} \leqslant (n!)^{-1}$. At this point, we also need to make use of the assumption $\rho^{\mu/2} \leqslant 1/8$ which illustrates that our method does not yield uniform control over $\mathscr{R}$ as $\mu \to 0$. Further details on this estimate can be found in [5]. Similarly, one proves the bound on $\|\partial_r w_{M,N}\|_{\Delta}^{(1)}$. This completes the proof of Theorem V.8. ∎

Theorem V.8 implies that,

$$\mathscr{R}_\rho\colon \mathscr{B}(\delta, \varepsilon) \to \mathscr{B}(\delta+\eta\varepsilon, \eta\varepsilon), \tag{V.120}$$

provided that $\delta \leqslant 1/8$, $\varepsilon\rho^{-1/2}$ is sufficiently small, and $\eta = 8\rho^{\mu/2} \leqslant 1/2$, as specified in (V.99). Applying this relation iteratively, we arrive at

THEOREM V.9. *Let $\mu > 0$ and pick ρ and ξ satisfying $0 < \rho^\mu \leqslant \frac{1}{16}$ and $0 < 2\sqrt{\pi}\,\xi \leqslant \min\{1/12, \rho^{(3+\mu)/4}\}$. Assume furthermore that $0 < \delta \leqslant 1/8$ and $\varepsilon\rho^{-1/2} \leqslant 1/12800$. Then, for all $n \geqslant 1$, we have*

$$\mathscr{R}_\rho^n\colon \mathscr{B}(\delta, \varepsilon) \to \mathscr{B}(\delta_{(n)}, \eta^n \varepsilon), \tag{V.121}$$

where $\delta_{(n)} = \delta + \varepsilon \sum_{k=1}^n \eta^k$.

Let us define $\rho_0 := g^{2-2\tau} \leqslant \min\{1, \frac{1}{2}\sqrt{1-\cos\vartheta}\,\delta\}$ as in Chapter IV, Eq. (IV.27), for some $0 < \tau < 1$. We note that Hypothesis 3 implies that Hypothesis 2 holds for any $0 \leqslant \beta \leqslant 1$. Our initial condition for the iteration of $\mathscr{R}$ is given by $H_{(0)} = (E_{(0)}, T_{(0)}, \underline{W}_{(0)})$. Upon the choice

$$\begin{aligned} &\rho_0 \leqslant g^{3/2}, && \varepsilon_{(0)} := C g^{1/2}, \\ &\rho := \min\{4\pi, 2^{-16/\mu}\}, \qquad \text{and} && 2\sqrt{\pi}\,\xi := \min\{1/12, \rho^{(3+\mu)/4}\}, \end{aligned} \tag{V.122}$$

it is ensured that $H_{(0)} \in \mathscr{B}(\varepsilon_{(0)}, \rho\varepsilon_{(0)})$ (see Theorem V.8 and further details in [5]).

V.4. *Spectrum of $H_g(\theta)$*

In this section, we locate the spectrum of $H_g(\theta)$ by means of the isospectral property of $\mathscr{R}_\rho$.

V.4.1. *Cuspidal domains of spectrum.* First, we investigate the convergence of the composition of the maps $\mathscr{Z}_{(n)}$, for $n \geqslant 0$. Recall from (V.42) that the iterated application of $\mathscr{R}_\rho$ onto $H_{(0)}[z]$ is defined as $H_{(n)}[z] \equiv (E_{(n)}[z], T_{(n)}[z], W_{(n)}[z]) = \mathscr{R}_\rho^n(H_{(0)})[z]$. The isospectral property of $\mathscr{R}_\rho$ guarantees that

$$\mathscr{Z}_{(n)}^{-1}[z] \in \sigma_\#(H_g(\theta)) \Leftrightarrow 0 \in \sigma_\#(H_{(n)}[z] - z), \tag{V.123}$$

where

$$\mathscr{Z}_{(n)}^{-1} = Z_{(0)}^{-1} \circ Z_{(1)}^{-1} \circ \cdots \circ Z_{(n)}^{-1}\ : D_{1/2} \to \mathscr{U}_{(n)}^{(\mathrm{in})}, \tag{V.124}$$

with $Z_{(0)}(\zeta) = e^{i\vartheta} \rho_0^{-1}(\zeta - E_j)$ on $\mathscr{U}_{(0)}^{(\mathrm{in})} := D(E_j, \rho_0/2)$, and, for $n \geqslant 1$,

$$Z_{(n)}\colon \mathscr{U}_{(n)}^{(\mathrm{in})} \to D_{1/2}, \qquad \zeta \mapsto \frac{1}{\rho}(\zeta - E_{(n-1)}[\zeta]), \tag{V.125}$$

where

$$\mathscr{U}_{(n)}^{(\mathrm{in})} = \{\zeta \in D_{1/2} \mid |\zeta - E_{(n-1)}[\zeta]| \leqslant \rho/2\} \tag{V.126}$$

(cf. Eqs. (V.43)–(V.46)). We define

$$\mathscr{S}_{(n)} = \mathscr{Z}_{(n)}^{-1}(D_{1/4}). \tag{V.127}$$

For $A \subseteq \mathbb{C}$, the inner and outer radius of A are defined by

$$\text{inner rad}(A) := \sup_{z,\, r} \{r \mid D(z, r) \subseteq A\} \qquad \text{and} \tag{V.128}$$

$$\text{outer rad}(A) := \inf_{z,\, r} \{r \mid D(z, r) \supseteq A\}. \tag{V.129}$$

LEMMA V.10. *For all $n = 1, 2, ...$*

$$\mathscr{S}_{(0)} \supseteq \mathscr{S}_{(1)} \supseteq \mathscr{S}_{(2)} \supseteq \cdots \supseteq \mathscr{S}_{(n)}, \tag{V.130}$$

$$\rho_0 \left(\frac{2\rho}{5}\right)^n \leqslant \text{inner rad}(\mathscr{S}_{(n)}) \leqslant \text{outer rad}(\mathscr{S}_{(n)}) \leqslant \rho_0 \left(\frac{4\rho}{3}\right)^n. \tag{V.131}$$

where the number $E_j(g) \in \mathbb{C}$ defined by $\{E_j(g)\} = \bigcap_{n \in \mathbb{N}} \mathscr{S}_{(n)}$, is uniquely determined by the sequence

$$E_j(g) = \lim_{n \to \infty} \mathscr{Z}_{(n)}^{-1}(0). \tag{V.132}$$

Proof. Since, $\mathscr{U}_{(n)}^{(\mathrm{in})} \subseteq D_{1/4}$, for $n = \mathbb{N}_0$, we clearly have

$$\mathscr{S}_{(n+1)} = \mathscr{Z}_{(n+1)}^{-1}(D_{1/4}) \subseteq \mathscr{Z}_{(n+1)}^{-1}(D_{1/2})) \subseteq \mathscr{Z}_{(n)}^{-1}(D_{1/4}) = \mathscr{S}_{(n)}, \tag{V.133}$$

and thus (V.130). By the same argument, $\mathscr{Z}_{(n)}^{-1}(\mathscr{U}_{(n)}^{(\mathrm{in})}) \subseteq \mathscr{Z}_{(n-1)}^{-1}(\mathscr{U}_{(n-1)}^{(\mathrm{in})}) \subseteq \cdots \subseteq \mathscr{Z}_{(0)}^{-1}(\mathscr{U}_{(0)}^{(\mathrm{in})})$, and $|\partial_z E_{(n)}| \leqslant 4\varepsilon\eta^n \leqslant 1/4$, by Theorem V.9. Thus, for $z \in \mathscr{U}_{(n)}^{(\mathrm{in})}$,

$$\frac{3}{4\rho}\,|z| \leqslant |Z_{(n)}(z)| = \rho^{-1}\,|z - E(z)| \leqslant \frac{5}{4\rho}\,|z|, \tag{V.134}$$

which implies that

$$\frac{4\rho}{5}\,|z| \leqslant |Z_{(n)}^{-1}(z)| \leqslant \frac{4\rho}{3}\,|z|, \tag{V.135}$$

for $z \in D_{1/2}$. Iterating this estimate, we obtain

$$\text{outer rad}\{\mathscr{Z}_{(n)}^{-1}(D_{1/4})\} \leqslant \frac{\rho_0}{4} \cdot \left(\frac{4\rho}{3}\right)^n, \tag{V.136}$$

proving the right inequality in (V.131). Next, using $|\partial_z E_{(n)}| \leqslant 1/4$ again, we infer that, for $z \in D \setminus \{0\}$,

$$|\arg[\mathscr{Z}_{(n)}^{-1}(z)] - \arg[z]| \leqslant \pi/4, \tag{V.137}$$

and hence

$$\text{inner rad}\{\mathscr{Z}_{(n)}^{-1}(D_{1/4})\} \geqslant \frac{\rho_0}{4}\left(\frac{2\rho}{5}\right)^n. \quad \blacksquare \tag{V.138}$$

Having found $E_j(g)$, the number in $\mathbb{C}_-$ that we later identify to be the resonance we sought for, we also wish to determine a deformed line segment, i.e., a function,

$$T_j(g, \cdot) \equiv T_{(\infty)}: [0, 1] \to \mathbb{C}_-, \tag{V.139}$$

that represents the "continuous spectrum" for the perturbed operator $H_g(\theta)$. We put "continuous spectrum" in quotation marks because we do not prove the existence of continuous spectrum for $H_g(\theta)$, but we rather show that any spectrum of $H_g(\theta)$ in $\mathscr{U}_{(0)}^{(\text{in})} = E_0 + D_{\rho_0/2}$ is contained in a cuspidal domain about $E_j(g) + \{T_{(\infty)}(r) \mid r \in [0, 1]\}$. In fact, since $H_g(\theta)$ is not self-adjoint, the notion of a spectral measure may not make sense for $H_g(\theta)$ at all.

We outline the construction of $T_{(\infty)}$, and we refer the reader to [5] for details. First, we define functions

$$\zeta_{(n)}: [0, 5/16] \to D_{3/8}, \tag{V.140}$$

for each $n \in \mathbb{N}_0$, by the requirement that $|\zeta_{(n)}(r) - r| \leqslant 1/16$ and that

$$\zeta_{(n)}(r) = T_{(n)}[\zeta_{(n)}(r); r], \tag{V.141}$$

where $T_{(n)}$ is defined in (V.42). To see that Eq. (V.141) has a unique solution for every $r \in [0, 5/16]$, we set $\zeta_{(n)}(r) := r + \delta$ and $\Delta T(\tau) := T_{(n)}[r + \tau; r] - r$. Then (V.141) reads

$$\delta = \Delta T(\delta). \tag{V.142}$$

By Cauchy's estimate and $\varepsilon \leqslant 1/16$, we have that

$$|\partial_z \Delta T(\zeta)| \leqslant \varepsilon \cdot (\tfrac{1}{2} - \tfrac{5}{16} - |\zeta|)^{-1} \leqslant \tfrac{1}{2}, \tag{V.143}$$

for $|\zeta| \leqslant 1/16$, and the existence of δ in (V.142) follows from a fix point argument, indeed.

Secondly, we pick $\phi \in C^\infty(\mathbb{R}_{+0})$ such that $\phi' \leqslant 0$, $\phi \equiv 1$ on $[0, 1/4]$, and $\phi \equiv 0$ on $[5/16, \infty]$. We use ϕ to define, for $n = 0, 1, 2, 3...$

$$\zeta_{(n)}^{(av)}(r) := \phi(r) \cdot \zeta_{(n)}(r) + (1 - \phi(r)) \cdot Z_{(n)}[\zeta_{(n-1)}(\rho r)], \qquad \text{(V.144)}$$

where $\zeta_{(-1)}(r) := r$ and $0 \leqslant r < 1$.

Next, we set

$$E_j(g) + T_{(\infty)}(r) := \sum_{n=0}^{\infty} \chi[\rho^{n+1} \leqslant r < \rho^n] \cdot \mathscr{Z}_{(n)}^{-1}[\zeta_{(n)}^{(av)}(\rho^{-n} r)], \qquad \text{(V.145)}$$

for all $r \in [0, 1]$. Although this is not obvious, $T_{(\infty)}$, as defined in (V.145), is Lipshitz continuous.

Finally, we define a cuspidal domain, for $\tau > 0$,

$$\mathscr{K}_{(\infty)}(\tau) := \{ T_{(\infty)}(r) + b \mid 0 \leqslant r < 1,\ |b| \leqslant \tau \cdot r^{1 + (\mu/4)} \}, \qquad \text{(V.146)}$$

and we claim that the following inclusion holds true.

THEOREM V.11. *Assume that* (V.122) *holds and let* $g\rho_0^{-1/2}$ *be sufficiently small. Then, there exists a constant, C, such that the spectrum of* $H_g(\theta)$ *obeys*

$$\sigma(H_g(\theta)) \cap \mathscr{U}_{(0)}^{(\text{in})} \subseteq E_j(g) + \mathscr{K}_{(\infty)}(C\varepsilon_{(0)}\rho^{-9/2}). \qquad \text{(V.147)}$$

We remark that Theorem V.11 implies (I.51) in Theorem I.3.

We sketch some key ideas in the proof of TheoremV.11. First, we use the functions $\zeta_{(n)}^{(av)}$ from (V.144) to define, for any $\delta' > 0$,

$$\mathscr{U}_{(n)}^{(\text{out})}(\delta') := \{ z \in D_{1/2} \backslash \mathscr{U}_{(n)}^{(\text{in})} \mid \forall r\colon |z - \zeta_{(n)}^{(av)}(r)| \geqslant \delta' \cdot |z - E_{(n)}(z)| \} \qquad \text{(V.148)}$$

(see Fig. V.4).

Then we show that a sufficiently large choice of δ' ensures that $\mathscr{U}_{(n)}^{(\text{out})}(\delta')$ is contained in the resolvent set of $H_{(n)}$. More precisely, we have

THEOREM V.12. *Then there is a constant* $C \geqslant 0$ *such that, for all* $n \geqslant 1$, $H_{(n)}(z) - z$ *is invertible, for all* $z \in \mathscr{U}_{(n)}^{(\text{out})}(C\varepsilon\rho^{-2}\eta^{n-1})$.

Using the isospectral property of the Feshbach map, we conclude from Theorem V.12 that the resolvent set, $\rho(H_g(\theta))$, of the dilated Hamiltonian contains

$$\mathscr{U}_{(0)}^{(\text{in})} \cap \rho(H_g(\theta)) \supseteq \bigcup_{n=0}^{\infty} \mathscr{Z}_{(n)}^{-1}[\mathscr{U}_{(n+1)}^{(\text{out})}(C\varepsilon\rho^{-2}\eta^n)]. \qquad \text{(V.149)}$$

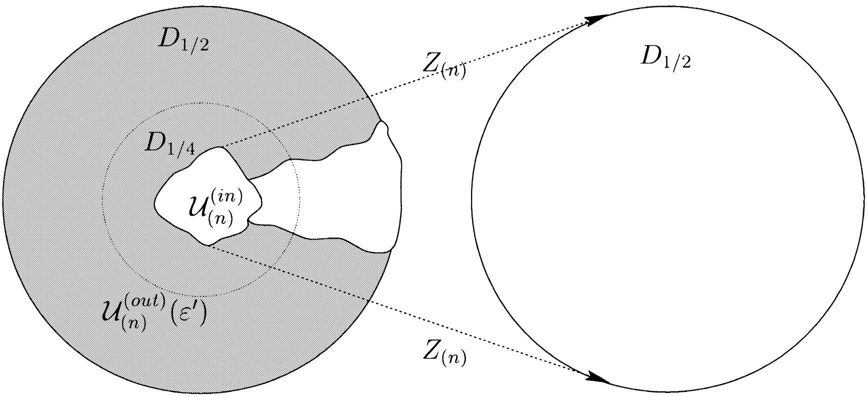

FIG. V.4. Identifying the resolvent set by means of $\mathscr{U}^{(\mathrm{out})}_{(n)}$.

Choosing $\delta'_n := 108\varepsilon\rho^{-2}(8\rho^{\mu/2})^{n-1}$, we obtain from (V.149) that

$$\sigma(H_g(\theta)) \cap \mathscr{U}^{(\mathrm{in})}_{(0)} \subseteq \mathscr{U}^{(\mathrm{in})}_{(0)} \Big\backslash \bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}[\mathscr{U}^{(\mathrm{out})}_{(n+1)}(\delta'_{n+1})]$$

$$= \left(\bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}[D_{1/2} \backslash \mathscr{U}^{(\mathrm{in})}_{(n+1)}] \right) \Big\backslash \left(\bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}[\mathscr{U}^{(\mathrm{out})}_{(n+1)}(\delta'_{n+1})] \right)$$

$$= \bigcup_{n=1}^{\infty} \mathscr{Z}^{-1}_{(n-1)}[D_{1/2} \backslash (\mathscr{U}^{(\mathrm{in})}_{(n)} \cup \mathscr{U}^{(\mathrm{out})}_{(n)}(\delta'_n))], \tag{V.150}$$

additionally using the fact that $\mathscr{U}^{(\mathrm{out})}_{(n)}(\delta'_n) \subseteq D_{1/2} \backslash \mathscr{U}^{(\mathrm{in})}_{(n)}$ and the pairwise disjointness of $Z^{-1}_{(n)}[D_{1/2} \backslash \mathscr{U}^{(\mathrm{in})}_{(n+1)}] \subseteq \mathscr{U}^{(\mathrm{in})}_{(0)}$, for different values of $n \in \mathbb{N}$. Then, we obtain Theorem V.11 from (V.150) by showing that, for every $n \in \mathbb{N}$,

$$\mathscr{Z}^{-1}_{(n-1)}[D_{1/2} \backslash (\mathscr{U}^{(\mathrm{in})}_{(n)} \cup \mathscr{U}^{(\mathrm{out})}_{(n)}(\delta'_n))] \subseteq E_j(g) + \mathscr{K}_{(\infty)}(C\varepsilon_{(0)}\rho^{-9/2}). \tag{V.151}$$

V.5. *Existence of Resonances*

Our last topic is the proof of the existence of a resonance in the vicinity of E_j. More precisely, we now prove

THEOREM V.13. *Let $E_j(g)$ be the number constructed in Lemma* V.10 *and assume that g is sufficiently small. Then $E_j(g)$ is an eigenvalue of $H_g(\theta)$ with normalized eigenvector, $\psi_g(\theta) \in H_{el} \otimes \mathscr{F}$, that has a non-vanishing overlap with $\varphi_{el,j} \otimes \Omega$.*

Proof. The key element of our proof consists in estimating the overlap of $\psi_g(\theta)$ with $\varphi_{el,j} \otimes \Omega$. We obtain $\psi_g(\theta)$ as a limit of a sequence, $\{\psi_m\}_{m>0}$,

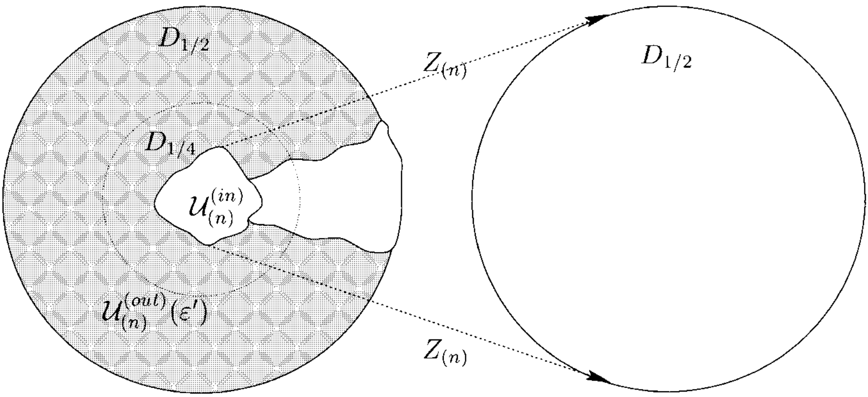

FIG. V.4. Identifying the resolvent set by means of $\mathcal{U}^{(\mathrm{out})}_{(n)}$.

Choosing $\delta'_n := 108\varepsilon\rho^{-2}(8\rho^{\mu/2})^{n-1}$, we obtain from (V.149) that

$$\begin{aligned}\sigma(H_g(\theta)) \cap \mathcal{U}^{(\mathrm{in})}_{(0)} &\subseteq \mathcal{U}^{(\mathrm{in})}_{(0)} \Big\backslash \bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}[\mathcal{U}^{(\mathrm{out})}_{(n+1)}(\delta'_{n+1})] \\ &= \left(\bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}[D_{1/2} \backslash \mathcal{U}^{(\mathrm{in})}_{(n+1)}]\right) \Big\backslash \left(\bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}[\mathcal{U}^{(\mathrm{out})}_{(n+1)}(\delta'_{n+1})]\right) \\ &= \bigcup_{n=1}^{\infty} \mathscr{Z}^{-1}_{(n-1)}[D_{1/2} \backslash (\mathcal{U}^{(\mathrm{in})}_{(n)} \cup \mathcal{U}^{(\mathrm{out})}_{(n)}(\delta'_n))], \end{aligned} \tag{V.150}$$

additionally using the fact that $\mathcal{U}^{(\mathrm{out})}_{(n)}(\delta'_n) \subseteq D_{1/2} \backslash \mathcal{U}^{(\mathrm{in})}_{(n)}$ and the pairwise disjointness of $Z^{-1}_{(n)}[D_{1/2} \backslash \mathcal{U}^{(\mathrm{in})}_{(n+1)}] \subseteq \mathcal{U}^{(\mathrm{in})}_{(0)}$, for different values of $n \in \mathbb{N}$. Then, we obtain Theorem V.11 from (V.150) by showing that, for every $n \in \mathbb{N}$,

$$\mathscr{Z}^{-1}_{(n-1)}[D_{1/2} \backslash (\mathcal{U}^{(\mathrm{in})}_{(n)} \cup \mathcal{U}^{(\mathrm{out})}_{(n)}(\delta'_n))] \subseteq E_j(g) + \mathscr{K}_{(\infty)}(C\varepsilon_{(0)}\rho^{-9/2}). \tag{V.151}$$

V.5. *Existence of Resonances*

Our last topic is the proof of the existence of a resonance in the vicinity of E_j. More precisely, we now prove

THEOREM V.13. *Let $E_j(g)$ be the number constructed in Lemma* V.10 *and assume that g is sufficiently small. Then $E_j(g)$ is an eigenvalue of $H_g(\theta)$ with normalized eigenvector, $\psi_g(\theta) \in H_{el} \otimes \mathscr{F}$, that has a non-vanishing overlap with $\varphi_{el,j} \otimes \Omega$.*

Proof. The key element of our proof consists in estimating the overlap of $\psi_g(\theta)$ with $\varphi_{el,j} \otimes \Omega$. We obtain $\psi_g(\theta)$ as a limit of a sequence, $\{\psi_m\}_{m>0}$, where $\lambda^m_{(n-1)} := (\hat{Z}^m_{(n)})^{-1} [\lambda^m_{(n)}]$ and

$$\begin{aligned}\tilde{\psi}^m_{(n-1)} = (\chi_\rho + \bar{\chi}_\rho [\bar{\chi}_\rho H^m_{(n-1)}(\lambda^m_{(n-1)}) \bar{\chi}_\rho - \lambda^m_{(n-1)}]^{-1} \bar{\chi}_\rho W^m_{(n-1)} \chi_\rho) \\ \times U_f[-\ln \rho]^* \tilde{\psi}^m_{(n)},\end{aligned} \tag{V.161}$$

where $\tilde{\psi}^m_{(n)} = \Omega$. By Theorem V.9 and Lemma V.1, we have that

$$\|\bar{\chi}_\rho [\bar{\chi}_\rho H^m_{(n-1)} \bar{\chi}_\rho - \lambda^m_{(n-1)}]^{-1} \bar{\chi}_\rho W^m_{(n-1)} \chi_\rho\| \leqslant \eta^n \varepsilon^{(0)} \rho^{-1/2}. \tag{V.162}$$

Thus

$$\|\tilde{\psi}^m_{(n-1)}\| \leqslant (1 + \varepsilon^{(0)} \rho^{-1/2} \eta^{n-1}) \cdot \|\tilde{\psi}^m_{(n)}\|. \tag{V.163}$$

Proceeding recursively, we obtain from (V.161) and (V.163) an eigenvector $\tilde{\psi}^m$ of H^m, corresponding to the eigenvalue $\lambda^m := (\mathscr{Z}^m_{(n)})^{-1} [\lambda^m_{(n)}]$, whose norm is bounded by

$$\|\tilde{\psi}^m\| \leqslant \prod_{j=0}^{n} (1 + \varepsilon^{(0)} \rho^{-1/2} \eta^{n-1}) \leqslant \exp\left[\frac{C\varepsilon^{(0)}}{\rho^{1/2}} \left(\frac{1}{1-\eta} \right) \right] < \infty. \tag{V.164}$$

Moreover, $|\langle \tilde{\psi}^m | \varphi_{el,j} \otimes \Omega \rangle| = 1$. Passing to $\psi^m := \|\tilde{\psi}^m\|^{-1} \tilde{\psi}^m$, we conclude that H^m has the eigenvalue λ^m with normalized eigenvector ψ^m. This eigenvector has a non-vanishing overlap

$$|\langle \psi^m | \varphi_{el,j} \otimes \Omega \rangle| \geqslant \exp\left[\frac{-C\varepsilon^{(0)}}{\rho^{1+(\mu/2)}} \left(\frac{1}{1-\eta} \right) \right] > 0, \tag{V.165}$$

uniformly in $m \to 0$. Now, we proceed in analogy to Section II.6. Indeed, we appeal to the proof of Theorem II.8 to establish the analogues of (II.30)–(II.34). Essentially, it remains to show that $m \mapsto H^m$ is a norm-resolvent continuous family of operators for $m \in [0, m_0]$ and $m_0 > 0$ sufficiently small. We define $\psi_g(\theta) := w-\lim_{m \to 0} \psi^m$ for a suitable subsequence $\{\psi^m\}_{m>0}$, noting that $\psi_g(\theta) \neq 0$, by (V.165). Thus, $\psi_g(\theta) \neq 0$ is the desired eigenvector of $H_g(\theta) = H_{m=0}$ since, by norm-resolvent continuity, $H_g(\theta)\, \psi_g(\theta) = E_j(g)\, \psi_g(\theta)$, for $E_j(g) := \lim_{m \to 0} \lambda^m$, indeed. ∎

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