# Resolution of Singularities I

## The Blow-Up Operation

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# 1. Set up.

$k :=$ field of characteristic 0

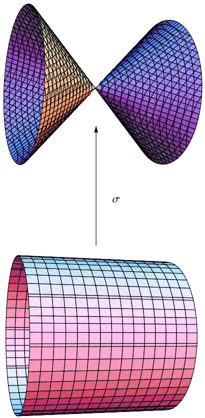
Defn. **Algebraic variety** $k^n \supset X :=$ zeros of $f_1, \ldots, f_k \in Pol$

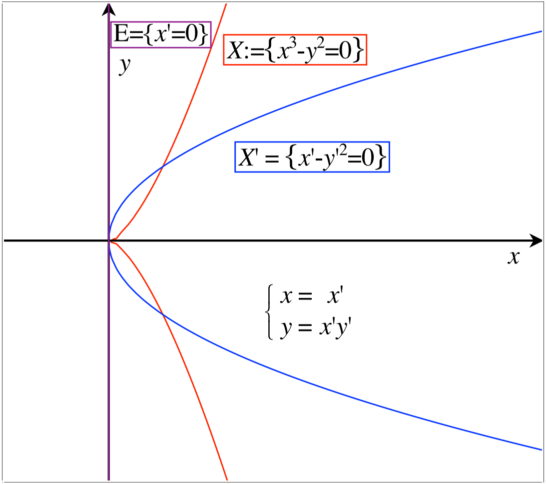
Defn. **Multiplicity** of $f$ at $a$ is

$$\mu_a(f) := \min \left\{ d : \frac{\partial^d f}{\partial x_{i_1}^{d_1} \ldots \partial x_{i_k}^{d_k}}(a) \neq 0 \text{ for } d_1 + \ldots + d_k = d \right\}$$

Defn. $a \in \mathrm{Sing}(V(f)) \iff \mu_a(f) > 1$

**e.g.** $Cone = V(f = x^2 - y^2 - z^2)$ $\quad \mathrm{Sing}(Cone) = \{0\}$ $\quad \mu_0(f) = 2$




For $(x, y) \in X, \quad (x, y) \mapsto (0, 0)$

Slope $y' = \frac{y}{x} \to 0 \iff \text{line}[x : y] \to x\text{-axis}$

# 2. Enticement.

Take "any" $f$ on manifold $M$ (f is in some fixed category, *e.g.* polynomials, or analytic functions)

Then $\exists$ proper morphism $\phi\colon M' \to M$

($\phi$ is a composition of "quadratic" maps)

$f \circ \phi$ is locally a monomial (up to invertible factor) on $M$

Furthermore, even $J(\phi) \cdot (f \circ \phi)$ is locally a monomial

# 3. Blowing-up: key instrument.

$C \hookrightarrow M$ manifolds. $U \subset M$ coordinate chart.

$C = V(I)$, ideal $I = (f_0, \ldots, f_m)$ on $U$.

$$[f]\colon U \backslash C \ni x \mapsto [f_0(x) : \ldots : f_m(x)] \in \mathbb{P}^m_{[\xi]}$$

$$\begin{array}{ccc} \mathrm{Bl}_I\, U := & \text{closure graph } [f] \subset & U \times \mathbb{P}^m \\ \downarrow\sigma & & \downarrow\pi \\ U & \xrightarrow{\quad\quad\quad \mathrm{id} \quad\quad\quad} & U \end{array}$$

closure graph $[f] = \{(x, [\xi])\colon \xi_j f_i(x) = \xi_i f_j(x)\}_{i,j \le m}$

So $\sigma\colon\ \mathrm{Bl}_I\, U \to U$ where $\sigma = \pi|_{\mathrm{Bl}_I\, U}$.

## 4.

$$U \times \mathbb{P}^m = \bigcup_j V'_j \qquad V'_j := \{\xi_j \neq 0\}. \qquad U'_j := V'_j \cap \mathrm{Bl}_I U.$$

Say $I := (x_1, \ldots, x_m)$ where $(x_1, \ldots, x_n)$ are coordinates on $U$.
So $C = V(I) = \{(x) \colon x_1 = \ldots = x_m = 0\}$.

$U'_j = V'_j \cap \{\xi_j x_i = \xi_i x_j\}_{i,j \leq m}$ So coordinates on $U'_j$ are:

$$\begin{aligned} y_j &= x_j \\ y_i &= \xi_i / \xi_j && \text{for } 1 \leq i \leq m, i \neq j \\ y_i &= x_i && \text{otherwise} \end{aligned}$$

*e.g.* on chart $U'_j$ $\qquad U'_j \backslash U'_i = \{y_i = 0\}$

and $\qquad U'_j \cap \sigma^{-1}(C) = \{y_j = 0\}$

Furthermore

$$\sigma|_{U'_j} \colon x_j = y_j \quad x_i = y_j y_i \ (1 \leq i \leq m, i \neq j) \quad x_i = y_i \ (\text{otherwise})$$

# 5. Example: $f := x^n - y^m = 0, \quad X = \{f = 0\} \subset k^2$

Assume $\gcd(n, m) = 1, n > m$.

$I = (x, y), \quad C = \{0\} = Sing(X)$

Fact: $\overline{\sigma^{-1}(X \backslash C)} \subset U'_x \qquad \sigma|_{U'_x} : (x_0, y_0) \mapsto (x_0, x_0 y_0)$

$f \circ \sigma(x_0, y_0) = x_0^n - (x_0 y_0)^m = x_0^m (x_0^{n-m} - y_0^m)$ for $(x_0, y_0) \in U'_x$

$f' := x_0^{-m}(f \circ \sigma) = x_0^{n-m} - y_0^m, \quad \sigma^{-1}(C) = \{x_0 = 0\}.$

**Summarizing.** $\sigma^{-1}(X) \cap U'_x = \{x_0 = 0\} \cup \{f' = 0\}$

Repeat. By Euclidean Algorithm:
Finitely many blowups $\implies X_n$ has no singularities.

# 6. Effect of blowing up.

$$M \supset X := \{x \colon f(x) = 0\} \qquad a \in X \qquad d := \mu_a(f)$$

Linear coordinate change $\implies a = 0$ and $\frac{\partial^d f}{\partial x_n^d}(a) \neq 0$ Why:

$f = \sum_{|\alpha| \geq d} c_\alpha x^\alpha \implies \sum_{|\alpha|=d} c_\alpha x^\alpha \neq 0, x \in \xi$ for some line $\xi$ Make $\xi$ into the $x_n$-axis.

$\tilde{x} := (x_1, \ldots, x_{n-1})$. Near $a$ can write

$$f(x) = c_0(\tilde{x}) + c_1(\tilde{x})x_n + \ldots + c_{d-1}(\tilde{x})x_n^{d-1} + c_d(x)x_n^d$$

$c_d(x) \neq 0$. Im.F.T. $\implies \frac{\partial^{d-1} f}{\partial x_n^{d-1}}(x) \sim (x_n - h(\tilde{x})) =: x_n'$ new coord.

$$N := \left\{ x : \frac{\partial^{d-1} f}{\partial x_n^{d-1}}(x) = 0 \right\}, \quad c_i := \frac{1}{i!} \cdot \frac{\partial^i f}{\partial x_n^i}\bigg|_N, \quad i < d$$

# 7. Blowing-up with $C = \{x_1 = \ldots = x_m = x_n = 0\}$

Two types of charts: $U'_n$ and all the others.

On $U'_n \setminus \bigcup_{j=1}^{m} U'_j = \{y_1 = \ldots = y_m = 0\}$ we show later $f' \neq 0$.

In coordinate chart $U'_j$ for $1 \leq j \leq m$. Say $j = 1$.

$\sigma^{-1}(C) \cap U'_1 = \{y_1 = 0\}$.

$c'_i := y_1^{i-d}(c_i \circ \sigma) \implies c'_i = c'_i(\tilde{y})$, for $i < d$
$\tilde{y} := (y_1, \ldots, y_{n-1})$

$$f' := y_1^{-d}(f \circ \sigma) = c'_0(\tilde{y}) + \ldots + c'_{d-2}(\tilde{y})y_n^{d-2} + c'_d(y)y_n^d$$

$c'_d(y) \neq 0 \; \forall y \in \sigma^{-1}(C) \cap U'_1$ since $\frac{\partial \sigma_n}{\partial y_n}(y) = y_1 = 0$
In particular $\mu_y(f') \leq d$.

## 8. On $U'_n \setminus \bigcup_{j=1}^m U'_j = \{y_1 = \ldots = y_m = 0\} \ni y$

$f'(y) := y_n^{-d}(f \circ \sigma)(y) = c'_0(y) + c'_1(y) + \ldots + c'_d(y)$,
where $c'_i := y_n^{i-d}(c_i \circ \sigma)$

$\mu_x(c_i) \geq d - i \; \forall x \in C, \quad i.e.$

$$c_i = \sum_{|\alpha| \geq d-i} c_{i_\alpha}(x_{m+1}, \ldots, x_{n-1}) \cdot x_1^{\alpha_1} \ldots x_m^{\alpha_m}$$

Recall $(x_1, \ldots, x_m) = y_n \cdot (y_1, \ldots, y_m) \implies$

$$\left.\begin{array}{l} c'_d(y) \neq 0 \\ c'_j(y) = 0 \end{array}\right\} \implies \mu_y(f') = 0 \leq d$$

**Corollary.** $d = \mu_a(f) \implies \mu_{a'}(f') \leq \mu_a(f) \; \forall a' \in \sigma^{-1}(a)$.

In fact, $f'(y) \neq 0$ for $y \in U'_n \setminus \bigcup_{j \neq n} U'_j$

Conclusion: we don't need to consider $U'_n$ chart.

# 9. Monomial assumption

**Assumption:** (proved later using induction on dimension)
$\exists \Omega \in \mathbb{Q}^{n-1}$ such that $c_i(\tilde{x})^{d!/(d-i)} = (\tilde{x}^{\Omega})^{d!} \cdot c_i^*(\tilde{x})$

Where $\tilde{x}^{\Omega} := x_1^{\Omega_1} \dots x_{n-1}^{\Omega_{n-1}}, \quad d!\Omega_i \in \mathbb{N}$ with some $c_i^*(\tilde{x}) \neq 0 \; \forall \tilde{x}$.

$S_{(f,d)} := \{x \colon \mu_x(f) = d\} = \{x \colon x_n = 0$ and $\mu_{\tilde{x}}(\tilde{x}^{\Omega}) \geq 1\} = \bigcup_J Z_J$

$Z_J := \{x_n = 0, x_j = 0 \; \forall j \in J\}$, for $J$ s.th. $\sum_{j \in J} \Omega_j \geq 1$

Suffices to consider $J$ s.th. $0 \leq \left( \sum_{j \in J} \Omega_j \right) - 1 < \Omega_k \; \forall k \quad (*)$

Choose $C = Z_J$ for any such $J$. Say $J = \{1, \dots, m\}$.

# 10. Multiplicity decreases.

We will show $|\Omega| := \sum_{k=1}^{n-1} \Omega_k$ decreases at points $y \in \sigma^{-1}(a)$ with $\mu_{a'}(f') = \mu_a(f) = d$. On $U'_j$ say $j = 1$

Reminder: corollary $\implies \mu_y(f') \leq d$.

Either multiplicity decreases (done) or multiplicities are equal.

$$
\begin{aligned}
c'_i(\tilde{y})^{d!/(d-i)} &= (y_1^{(i-d)d!/(d-i)})(\sigma(\tilde{y})^{\Omega})^{d!}(c_i^* \circ \sigma(\tilde{y})) \\
&= (y_1^{\left(\sum_{k\in J}\Omega_k\right)-1} y_2^{\Omega_2} \dots y_{n-1}^{\Omega_{n-1}})^{d!}(c_i^* \circ \sigma(\tilde{y}))
\end{aligned}
$$

Recall:
$f' = c'_0(\tilde{y}) + \ldots + c'_{d-2}(\tilde{y})y_n^{d-2} + c'_d(y)y_n^d \quad$ with $\quad c'_d(y) \neq 0 \;\forall y$

# 11. If $\mu_{a'}(f') = d$ then $y_n = 0$

$\Omega' := (\sum_{k=1}^{m} \Omega_k - 1, \Omega_2, \ldots, \Omega_{n-1})$

Then $(\tilde{y}^{\Omega'})^{d!} \mid (c_i')^{d!/(d-i)}\ \forall i$ with some $(c_i^* \circ \sigma)(\tilde{y}) \neq 0\ \forall \tilde{y}$
*i.e.* $f'$ satisfies monomial assumption

$|\Omega'| < |\Omega|$ by choice of $C$ (see inequality $(*)$) $\implies$
$(\mu_{a'}(f'), |\Omega'|) < (\mu_a(f), |\Omega|)$ (ordered lexicographically)

Conclusion: multiplicity $d$ decreases when $\sum_{k=1}^{n+1} \Omega_k < 1$.

After blowing up, $|\Omega'| - |\Omega| \geq 1/d!$ since $d!\Omega_i \in \mathbb{N}$

$\implies$ after at most $d!|\Omega|$ blowings-up, multiplicity decreases.

# 12. Induction on dim. $\implies$ monomial assumption

$A_f(\tilde{x}) := \prod'^{d-2}_{i=0} c_i^{d!/(d-i)}(\tilde{x}) \times$ ('their' differences)

$\prod'$ means include only nonzero factors

Desing. in $(n-1)$-dim $\implies$ $\exists \phi = \sigma_1 \circ \cdots \circ \sigma_\ell$ such that $(A_f \circ \phi)(\tilde{y})$ is locally monomial

So, W.L.O.G. may assume each factor in the product is locally a monomial on $N = \left\{ \frac{\partial^{d-1} f}{\partial x_n^{d-1}}(x) = 0 \right\}$.

Lemma (to come) $\implies$ exponents are totally ordered $\implies$ $\exists \Omega$ such that $(\tilde{x}^{\Omega})^{d!} \mid (c_i)^{d!/(d-i)}$

# 13. Lemma

$\alpha, \beta, \gamma \in \mathbb{N}^n \qquad a(x), b(x), c(x)$ invertible

$$a(x)x^\alpha - b(x)x^\beta = c(x)x^\gamma \implies \begin{cases} \text{either} & \alpha_i \le \beta_i \ \forall i \\ \text{or} & \beta_i \le \alpha_i \ \forall i \end{cases}$$

**Proof.** If $\alpha_k \le \beta_k \ \forall k$ done. Otherwise $\exists k$ s.t. $\beta_k < \alpha_k$.

Write $\alpha - \beta_k = (\alpha_1, \ldots, \alpha_k - \beta_k, \ldots, \alpha_n)$

$a(x)x^{\alpha-\beta_k} - b(x)x^{\beta-\beta_k} = c(x)x^{\gamma-\beta_k}$. Evaluate at $x_k = 0$

$$\implies c(x)x^{\gamma-\beta_k} = -b(x)x^{\beta-\beta_k} \neq 0 \implies \beta = \gamma$$

$$\implies a(x)x^\alpha = (b(x) + c(x))x^\beta.$$

If $\exists j$ s.t. $\alpha_j < \beta_j$, proceed similarly. Done $\blacksquare$

# 14. I cheated: used stronger induction than proved

I ignored exceptional factors that accumulated before the 'last' drop in multiplicity of $f \mapsto f' := y_{\mathrm{exc}}^{-d}(f \circ \sigma)$ (say 'old' exceptional divisors).

As a consequence, choices of centres might **not** have normal crossings with with 'old' exceptional divisors as well.

Say $E_{\mathrm{old}} := \{H_k \colon H_k \text{ old exc.}\}$

$E_{\mathrm{old}}(a) := \{H_k \in E_{\mathrm{old}} \colon a \in H_k\} \qquad s(a) := \# E_{\mathrm{old}}(a)$

$\{g_k = 0\} := H_k \quad \Longrightarrow \quad \mathrm{d}\, g_k \neq 0$

15.

Choose $x_n$ such that both

$$\frac{\partial^d f}{\partial x_n^d}(a) \neq 0 \quad \text{and} \quad \frac{\partial g_k}{\partial x_n}(a) \neq 0 \quad \forall H_k \in E_{\text{old}}(a)$$

Consider besides $c_j(\tilde{x}) = \left.\frac{\partial^j f}{\partial x_n^j}\right|_N, 0 \leq j \leq d-2$
all $a_k = g_k|_N$.

Now,

$$A_f(\tilde{x}) := {\prod}' c_j^{d!/(d-j)}(\tilde{x}) \times {\prod}' a_k^{d!}(\tilde{x}) \times (\text{'their' differences})$$

Before: $(\mu_{a'}(f'), |\Omega'|) < (\mu_a(f), |\Omega|)$ for $a' \in \sigma^{-1}(a)$

Now: $(\mu_{a'}(f'), s(a'), |\Omega'|) < (\mu_a(f), s(a), |\Omega|)$