Introductory Exercises II (MAT477 in 2009 - 2010) :

1 Bezout Theorem.

Let $\mathcal{P}_n := \mathbb{C}[x_1, \ldots, x_n]$, for any $a \in \mathbb{C}^n$ local ring $\mathcal{O}_a := \left\{ \frac{f}{g} : f, g \in \mathcal{P}_n,\ g(a) \neq 0 \right\}$ contains $\mathcal{P}_n$ (via $f \mapsto \frac{f}{1}$). For any ideal $I \subset \mathcal{P}_n$ similarly $\mathcal{O}_a / I \cdot \mathcal{O}_a$ contains $\mathcal{P}_n / I$. Denote $V(I) := \{ a \in \mathbb{C}^n : f(a) = 0 \quad \forall f \in I \}$. In the introductory lecture using Hilbert Nullstellensatz we proved by means of a version of the 'chinese remainder lemma'

Theorem 1.1. $\dim_{\mathbb{C}} \mathcal{P}_n / I < \infty$ *iff* $V(I) = \{a_1, \ldots, a_m\}$ *is finite and then natural map*

$$\mathcal{P}_n / I \to \bigoplus_{j=1}^{m} \mathcal{O}_{a_j} / I \cdot \mathcal{O}_{a_j}$$

is an isomorphism.

We will follow enumeration of exercises, notations and definitions of Introductory Exercises I (in particular below of the paragraph "Homogenization" of topic "Resultants").

Exercise 21. For a polynomial map $P : \mathbb{C}^n \to \mathbb{C}^n$ we define $\phi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ by formula

$$\phi(x_0, x) := (x_0, HP_1(x_0, x), \ldots, HP_n(x_0, x)), \quad (x_0, x) \in \mathbb{C}^{n+1} .$$

Show that $V(\mathcal{L}P_1, \ldots, \mathcal{L}P_n) = \{0\}$ iff $\phi^{-1}(0) = \{0\}$ and imply that maps P , ϕ and $\mathcal{L}P : \mathbb{C}^n \ni x \mapsto (\mathcal{L}P_1(x), \ldots, \mathcal{L}P_n(x)) \in \mathbb{C}^n$ are proper maps (in particular, the latter two 'are proper near $0 \in \mathbb{C}^{n+1}$ ' as well).

Exercise 22. Assume that $P^{-1}(0) = \{a_1, \ldots, a_m\}$ and that P has no solutions at infinity, i.e. $V(\mathcal{L}P_1, \ldots, \mathcal{L}P_n) = \{0\}$. Show that the mapping degrees $deg_0\ (\mathcal{L}P)$ and $deg_0\ (\phi)$ of ϕ (at $0 \in \mathbb{C}^n$ and, respectively, at $0 \in \mathbb{C}^{n+1}$ are well defined and) coincide with the sum over $j = 1 , \ldots , m$ of the (well defined) mapping degrees $deg_{a_j}\ (P)$ of P near a_j , $1 \leq j \leq m$.

Exercise 23. $\phi^{-1}(0) = \{0\}$ (cf. exercise 21.) implies

$$\mathcal{O}_0 / (\phi) \cdot \mathcal{O}_0 \cong \mathcal{O}_0 / (\mathcal{L}P_1, \ldots, \mathcal{L}P_n) \cdot \mathcal{O}_0 \cong \mathcal{P}_n / (\mathcal{L}P_1, \ldots, \mathcal{L}P_n) \cdot \mathcal{P}_n$$

Key Theorem 1. *(A possible topic to choose.)* $\phi^{-1}(0) = 0$ *implies*

$$deg_0\ (\phi) = \dim_{\mathbb{C}} \mathcal{O}_{\text{o}} / (\phi) \cdot \mathcal{O}_0 < \infty$$

Corollary 1.2. *Applied to* P *near* a_j *instead of* ϕ *near* 0 *it follows that*

$$deg_{a_j}\ (P) = \dim_{\mathbb{C}} \mathcal{O}_{\mathrm{a_j}}/(\mathrm{P}) \cdot \mathcal{O}_{\mathrm{a_j}}$$

and is refered to as the multiplicity of P *at* a_j *.*

Key Theorem 2. *(Another possible topic to choose.) With* ϕ *from Key Theorem 1,* $A(x)$ *an* $(n+1) \times (n+1)$ *matrix satisfying* $\phi(x) = A(x) \cdot x$ *and* $\mu := deg_0\ (\phi)$ *it follows*

$$\mu \cdot \det \mathrm{A(x)} = \det \frac{\partial \phi}{\partial \mathrm{x}}(\mathrm{x}) \neq 0 \textit{ in the ring } \mathcal{O}_0/(\phi) \cdot \mathcal{O}_0$$

Exercise 24. Applying the latter theorem to $\mathcal{L}P(x) := (\mathcal{L}P_1(x), \ldots, \mathcal{L}P_n(x))$ show that $V(\mathcal{L}P_1, \ldots, \mathcal{L}P_n) = \{0\}$ implies that

$$deg_0\ (\mathcal{L}P) = \prod_{i=1}^{n} \mathrm{degree(P_i)}$$

Corollary 1.3 (Bezout's Theorem)**.** *If* $P^{-1}(0)$ *has 'no solutions at infinity' (under inclusion* $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$*) then* $\#P^{-1}(0)$ *'counted with multiplicities' (i.e. all points of* $P^{-1}(0)$ *are counted with the multiplicities of* P *at the respective points) is the product* $\prod_{i=1}^{n} \mathrm{degree(P_i)}$ *of the polynomial degrees of* P_i*'s.*

Solution to Exercise 24. Set $h(s) := \mathcal{L}P(s \cdot x)$ for $s \in \mathbb{C}$. Then $h(1) - h(0) = \int_0^1 h'(s)\, ds$ and implies $\mathcal{L}P(x) = A(x) \cdot x$, where

$$A(x) = \left[\int_0^1 \frac{\partial \mathcal{L}P_i}{\partial x_j}(s \cdot x)\ ds \right]_{1 \leq i,j \leq n} .$$

Homogeneities of $\mathcal{L}P_i$'s of degrees d_i imply $\frac{\partial \mathcal{L}P_i}{\partial x_j}(s \cdot x) = s^{d_i - 1} \frac{\partial \mathcal{L}P_i}{\partial x_j}(x)$. Therefore

$$A(x) = \begin{pmatrix} \frac{1}{d_1} & 0 & \ldots \\ 0 & \frac{1}{d_2} & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & \frac{1}{d_n} \end{pmatrix} \cdot \left(\frac{\partial \mathcal{L}P}{\partial x}(x) \right)$$

Consequently $\det A(x) = \dfrac{1}{\prod_{i=1}^{n} d_i} \cdot \det \frac{\partial \mathcal{L}\mathrm{P}}{\partial \mathrm{x}}(\mathrm{x})$ and Key Theorem 2 implies $d_1 \cdots d_n = deg_0\ (\mathcal{L}P)$, while each $d_j = \mathrm{degree(P_j)}$, as required.

Exercise 25. Introduce a 'weighted' version of Bezout's Theorem: set $\mathrm{wt(x_j)} := \mathrm{w_j} \in \mathbb{Z}_+$ and $\mathrm{wt}(\mathrm{x}_1^{\alpha_1}, \ldots, \mathrm{x}_\mathrm{n}^{\alpha_\mathrm{n}}) := \sum_{\mathrm{j}=1}^{\mathrm{n}} \alpha_\mathrm{j} \mathrm{w_j}$. For $s \in \mathbb{C}$ and $x \in \mathbb{C}^n$ let $s * x := (s^{w_1} x_1, \ldots, s^{w_n} x_n)$. For $P \in \mathcal{P}_n - \{0\}$ weighted degree $\mathrm{wt.deg\, P}$ is the largest weighted degree of the monomials in its expansion. For any $P \in \mathcal{P}_n - \{0\}$ the 'weighted homogeneization' $H_{wt}P \in \mathcal{P}_{n+1}$ of P is $H_{wt}P(x_0, x) := x_o^{\mathrm{wt.deg\,P}} \cdot P(x_0^{-1} * x)$ and the 'leading weighted homogeneous form' $\mathcal{L}_{wt}P \in \mathcal{P}_n$ of P is $\mathcal{L}_{wt}P(x) := H_{wt}P(0, x)$. Using the same Key Theorems 1 and 2 show that $V(\mathcal{L}_{wt}P_1, \ldots, \mathcal{L}_{wt}P_n) = \{0\}$ implies

$$\prod_{j=1}^{n} \frac{\mathrm{wt.deg\,P_j}}{w_j} = \#P^{-1}(0) \quad \text{counted with multiplicities}$$

Calculus part of solution to Exercise 25. (Fill in the remaining details needed for a complete solution of Exercise 25.) Let $h(s) := \psi(s * x)$ with $\psi_i(x) := \mathcal{L}_{wt}P_i(x)$ being the components of ψ . Then $h(1) - h(0) = \int_0^1 h'(s)\, ds$ and implies $\psi(x) = A(x) \cdot x$, where

$$A(x) = \left[\int_0^1 w_j \cdot s^{w_j - 1} \cdot \frac{\partial \psi_i}{\partial x_j}(s * x)\ ds \right]_{1 \le i,j \le n} .$$

Weighted homogeneities of ψ_i's of weighted degrees d_i , i.e. $\psi_i(s * x) = s^{d_i} \cdot \psi_i(x)$, imply $\frac{\partial \psi_i}{\partial x_j}(s * x) = s^{(d_i - w_j)} \cdot \frac{\partial \psi_i}{\partial x_j}(x)$. Therefore,

$$A(x) = \begin{pmatrix} \frac{1}{d_1} & 0 & \ldots \\ 0 & \frac{1}{d_2} & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & \frac{1}{d_n} \end{pmatrix} \cdot \left(\frac{\partial \psi}{\partial x}(x) \right) \cdot \begin{pmatrix} w_1 & 0 & \ldots \\ 0 & w_2 & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & w_n \end{pmatrix}$$

Consequently $\det \mathrm{A(x)} = \left(\prod_{\mathrm{i=1}}^{\mathrm{n}} \frac{\mathrm{w_i}}{\mathrm{d_i}} \right) \cdot \det \frac{\partial \psi}{\partial \mathrm{x}}(\mathrm{x})$, Key Theorem 2 and $d_i = \mathrm{wt.deg\,P_i}$ imply

$$\prod_{i=1}^{n} \frac{\mathrm{wt.deg\,P_i}}{w_i} = \#P^{-1}(0) \quad \text{counted with multiplicities} ,$$

as required.