Bezout-type theorems

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Outline

Counting solutions of polynomial systems

Painters' perspective

Back to counting solutions

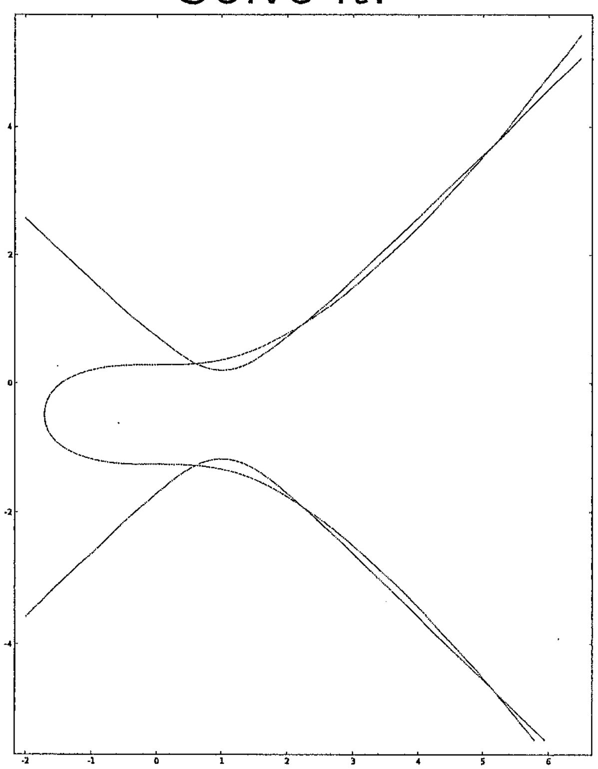
Outdoing the painters

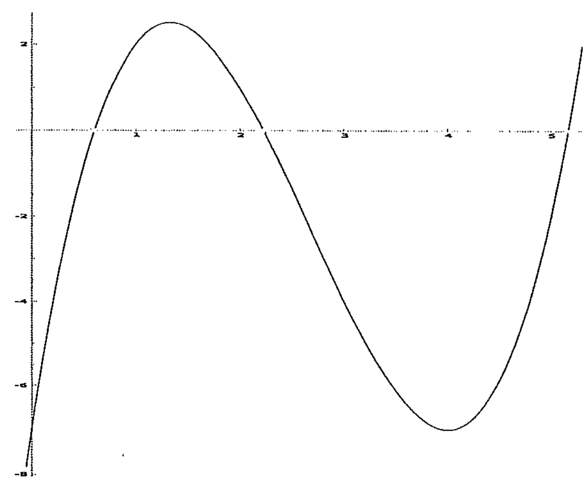
Warm up example

$$4y^2 - 4x^2 + 4y + 8x - 5 = 0 \quad (1) \qquad \times 2$$
$$x^3 - 8y^2 - 8y + 3 = 0 \quad (2)$$

$$\begin{array}{r} 8y^2 - 8x^2 + 8y + 16x - 10 = 0 \\ x^3 - 8y^2 - 8y + 3 = 0 \\ \hline x^3 - 8x^2 + 16x - 7 = 0 \end{array}$$

Solve it!





plug in solns of x into (2)
each $x \Rightarrow$ two solns for y
total 6 solutions

Bézout's Theorem

Theorem (Newton, 1660's (?), Bézout (1779))
Number of solutions of two polynomial curves is at most the product of their degrees.

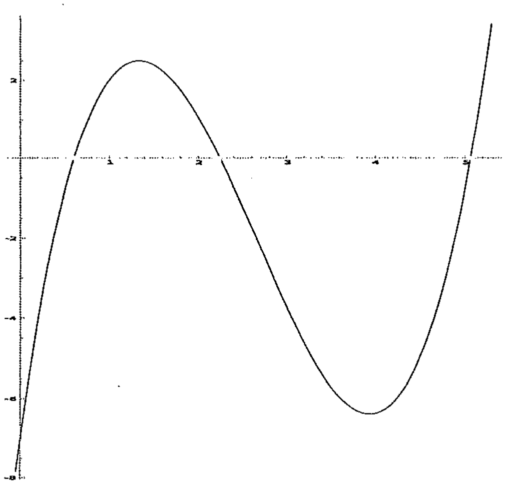
$$4y^2 - 4x^2 + 4y + 8x - 5 = 0 \qquad \Rightarrow \deg = 2$$
$$x^3 - 8y^2 - 8y + 3 = 0 \qquad \Rightarrow \deg = 3$$

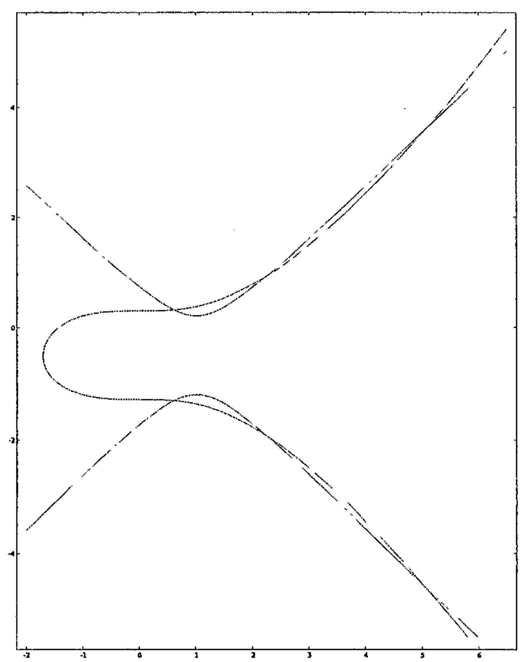
$\Rightarrow$ Bézout bound for number of solutions is $2 \times 3 = 6$

$\Rightarrow$ Bézout bound is exact!

Example 2: A variation on the warm up problem

$$
\begin{aligned}
f := & 0.005x^3 + 4y^2 - 4x^2 + 4y + 8x - 5 && = 0 \quad (1)\\
g := & x^3 - 8y^2 - 8y + 3 && = 0 \quad (2)\\
& 0.01x^3 + 8y^2 - 8x^2 + 8y + 16x - 10 && = 0 \quad (1) \times 2\\
& x^3 - 8y^2 - 8y + 3 && = 0\\
\hline
& 1.01x^3 - 8x^2 + 16x - 7 && = 0
\end{aligned}
$$





$\#$ solutions $= 6$

$\deg(f) = 3$, $\deg(g) = 3$

$\Rightarrow$ Bézout bound is $3 \times 3 = 9$

$\Rightarrow$ Not exact!

Weighted Bézout Theorem

Theorem (known at least since 1970's)

Number of solutions of $f = 0$ and $g = 0$ is less than or equal to

$$\frac{\mathrm{wt}(f)\,\mathrm{wt}(g)}{\mathrm{wt}(x)\,\mathrm{wt}(y)}.$$

$$\begin{aligned} f &= 0.005x^3 + 4y^2 - 4x^2\mathbf{0.005x^3} + 4y^2 - 4x^2 0.005x^3 + \mathbf{4y^2} - 4x^2\mathbf{0.(} \\ g &= x^3 - 8y^2\mathbf{x^3} - 8y^2x^3 - \mathbf{8y^2x^3} - \mathbf{8y^2} - 8y + 3 \end{aligned}$$

Assign different 'weights' to x and y

	wt(f)	wt(g)	wt. Bézout bound
$x \mapsto 1, y \mapsto 1$	3	3	$\frac{3\times3}{1\times1} = 9 > 6$
$x \mapsto 1, y \mapsto 2$	4	4	$\frac{4\times4}{1\times2} = 8 > 6$
$x \mapsto 2, y \mapsto 3$	6	6	$\frac{6\times6}{2\times3} = 6 = 6$

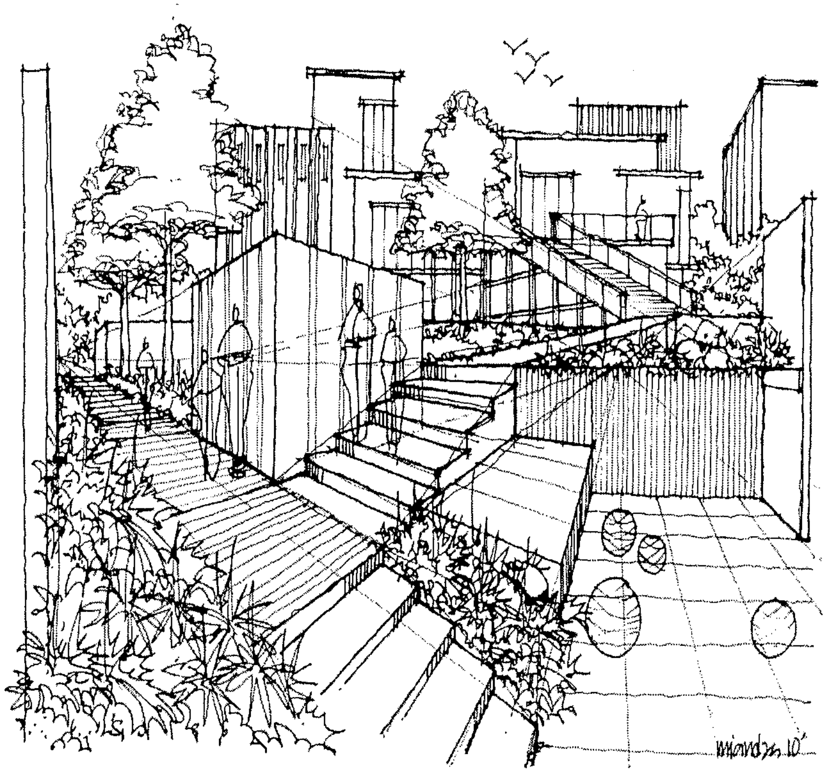
Wt. Bézout bound is exact for weights $(3, 2)$!

Different Perspectives

Perspective: parallel lines meet in a 'vanishing point'

Two point perspective: two vanishing points

Painting geometry

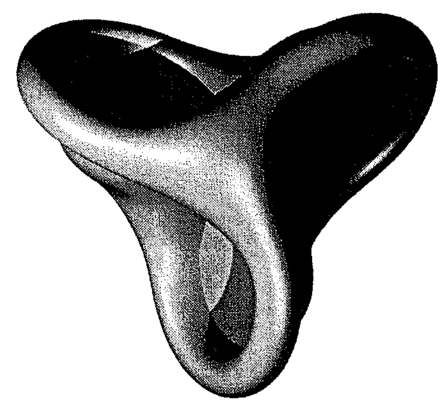


Euclidean Geometry: geometry of the 'Euclidean plane' $\mathbb{R}^2$.

Projective Geometry: add to $\mathbb{R}^2$ a 'line at infinity' which consists of vanishing points for all families of parallel lines.

The new space is the 'projective plane' $\mathbb{RP}^2$.

Multi-point perspective: every family of parallel line has a distinct vanishing point



Doing the same with the complex plane $\mathbb{C}^2$ gives rise to the complex projective plane $\mathbb{CP}^2$.

Back to counting solutions

Bézout's Theorem:

1. $\#\{f = g = 0\} \leq \deg(f)\deg(g)$.

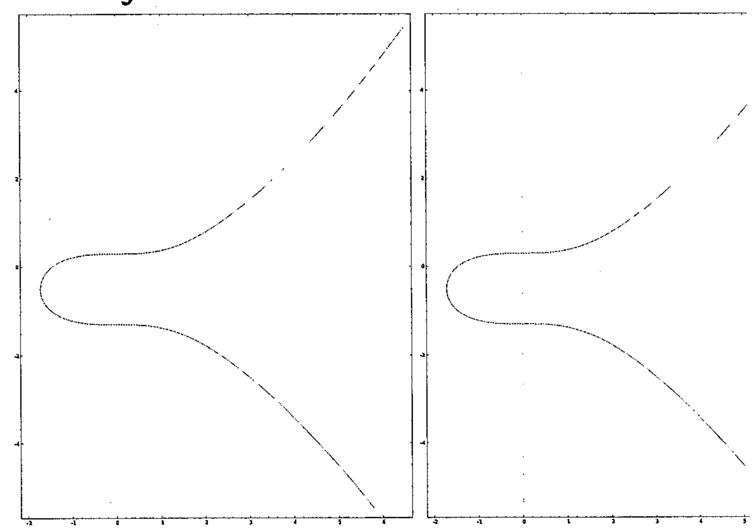
2. The bound is exact (for complex solutions) if the curves $f = 0$ and $g = 0$ 'approaches' different points at infinity on $\mathbb{CP}^2$.

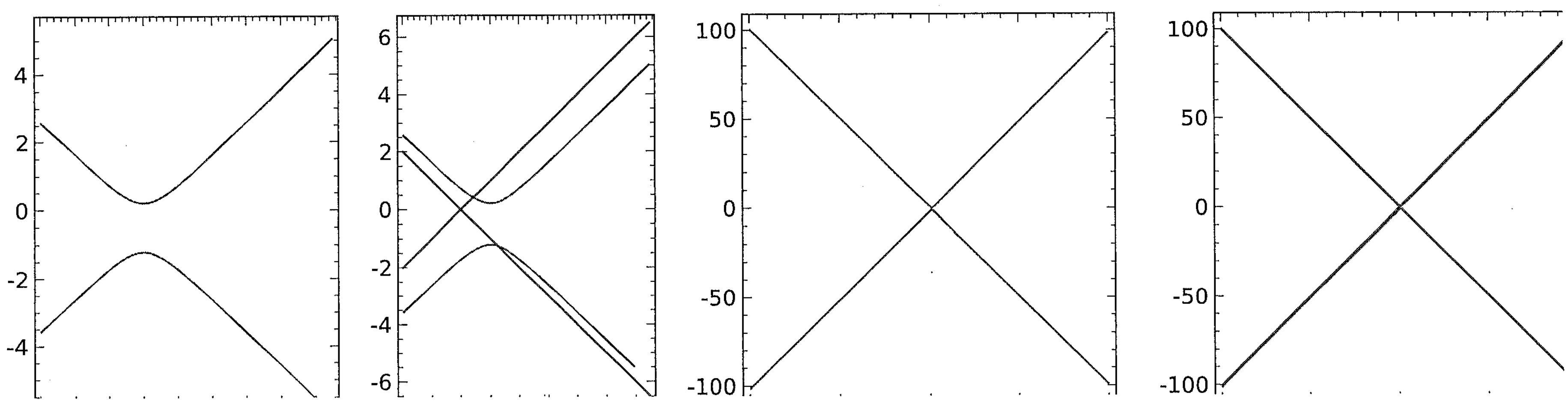
$g = x^3 - 8y^2 - 8y + 3g = \mathbf{x}^3 - 8y^2 - 8y + 3$

$g = 0$ meets the point at infinity on $\mathbb{CP}^2$ cor. to (lines parallel to) .

$f = 4y^2 - 4x^2 + 4y + 8x - 5f = \mathbf{4y^2} - \mathbf{4x^2} + 4y + 8x - 5$

$f = 0$ meets *two* points at infinity on $\mathbb{CP}^2$: cor. to $y = x$ and $y = -x$.





Example 2 revisited

Recall Example 2:

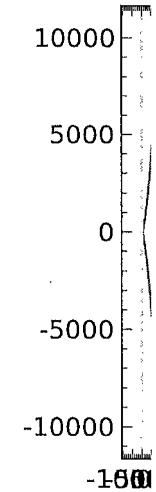
$f = 0.005x^3 + 4y^2 - 4x^2 + 4y + 8x - 5$

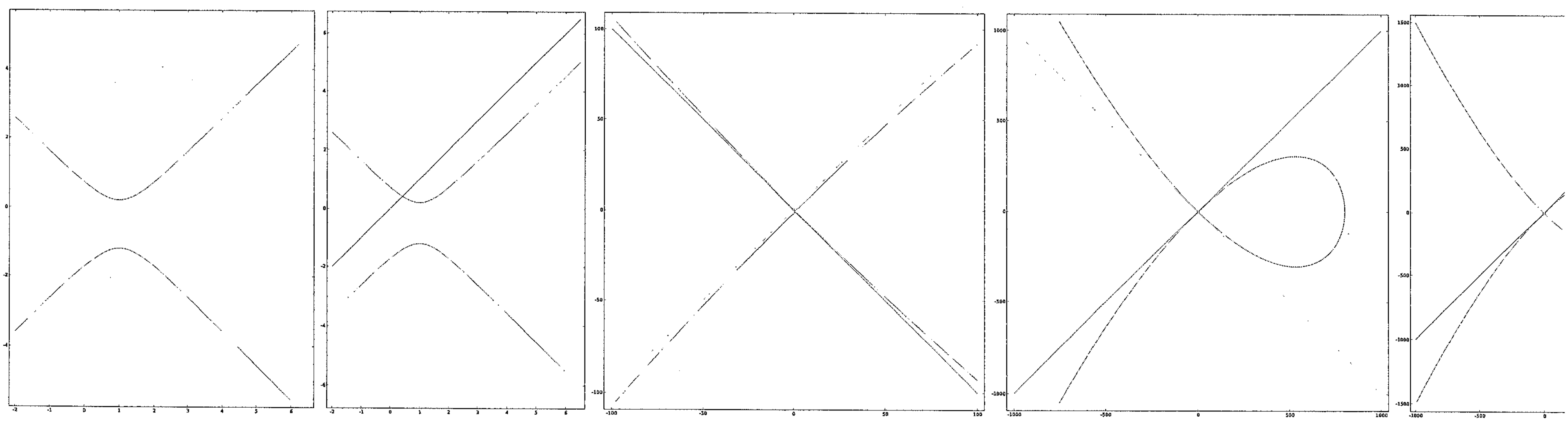
$g = x^3 - 8y^2 - 8y + 3$

$\#\{f = g = 0\} = 6 < 9 = \deg(f)\deg(g)$.

$g = 0$ meets the point at infinity on $\mathbb{CP}^2$ cor. to .

$f = 0$ meets the *same* point at infinity on $\mathbb{CP}^2$.

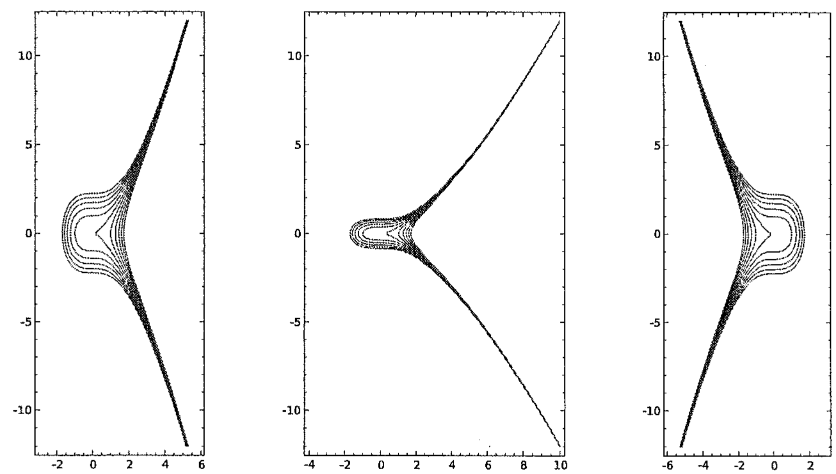




Weighted Projective Planes

Projective plane $\longleftrightarrow$ $\mathbb{R}^2$ + one point for each family of parallel lines $y = mx + c$

Weighted projective planes $\longleftrightarrow$ $\mathbb{R}^2$ + one point for each family of parallel curves $y^p = mx^q + c$ $(p, q$ fixed)



$p = 3,\ q = 2$

(We will deal with *complex* wt. projective spaces - just to make life a bit easier!)

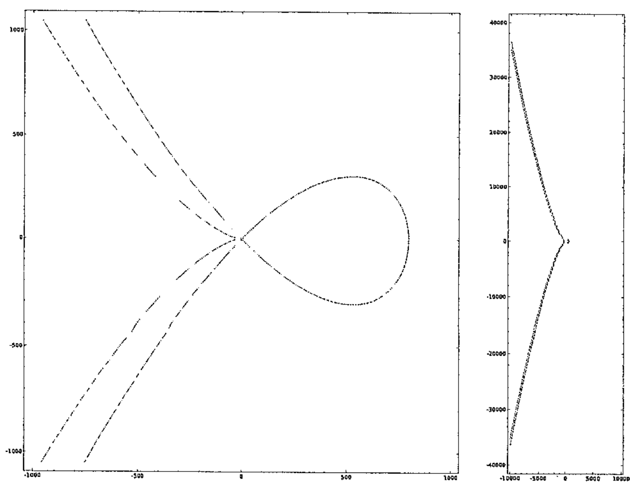
Example 2 again

Weighted Bézout Theorem: Consider weights $x \mapsto p$, $y \mapsto q$.

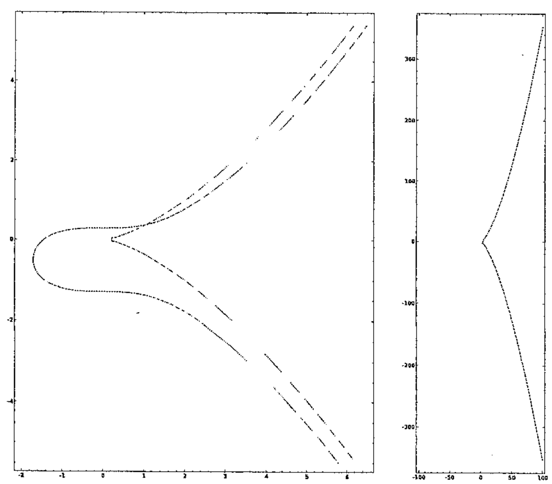
1. $\#\{f = g = 0\} \leq \dfrac{\mathrm{wt}(f)\,\mathrm{wt}(g)}{pq}$.
2. The bound is exact if the curves $f = 0$ and $g = 0$ 'approaches' different points at infinity on the cor. weighted projective plane.

Example 2: $\#\{f = g = 0\} = 6 = \dfrac{\mathrm{wt}(f)\,\mathrm{wt}(g)}{pq}$, $p = 2, q = 3$.

$$f = \quad 0.005x^3 + 4y^2 - 4x^2 + 4y + 8x - 5\mathbf{0.005x^3 + 4y^2} - 4x^2 + 4y +$$
$$g = \quad x^3 - 8y^2 - 8y + 3\mathbf{x^3} - \mathbf{8y^2} - 8y + 3$$



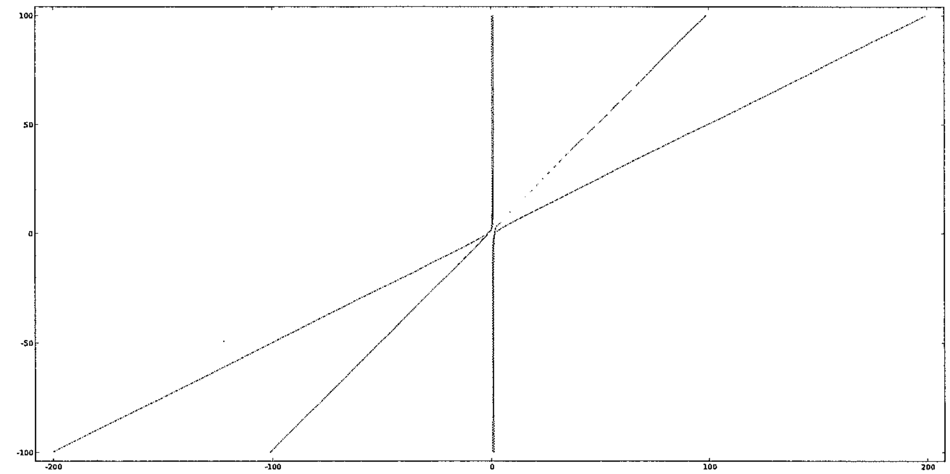
$f = 0$ 'approaches' the point at infinity cor. to $0.005x^3 + 4y^2 = 0$



$g = 0$ 'approaches' the point at infinity cor. to $x^3 - 8y^2 = 0$

What if weighted Bézout Theorem does not work?

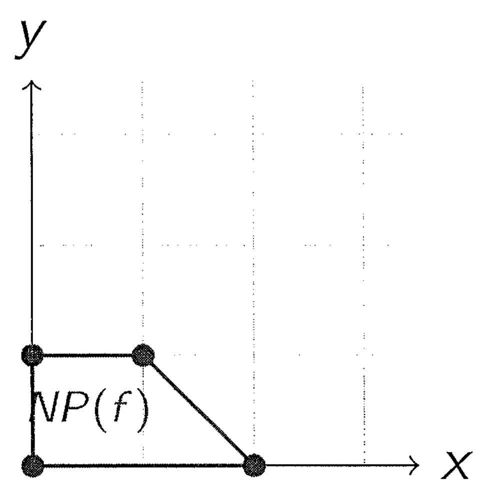
$f = x^2 - xy + y - 3$
$g = x^2 - 2xy - y - 4$
Has one common solution at infinity on $\mathbb{CP}^2$.
(Can be seen from *leading homogeneous forms*.)
Same is true for all weighted homogeneous planes.



For a better bound, consider the *Newton polygon* of f

Theorem (Kushnirenko, 1970s)

If f and g have the same Newton polygon $\mathcal{P}$, then $\#\{f = g = 0\} \leq 2Area(\mathcal{P})$.



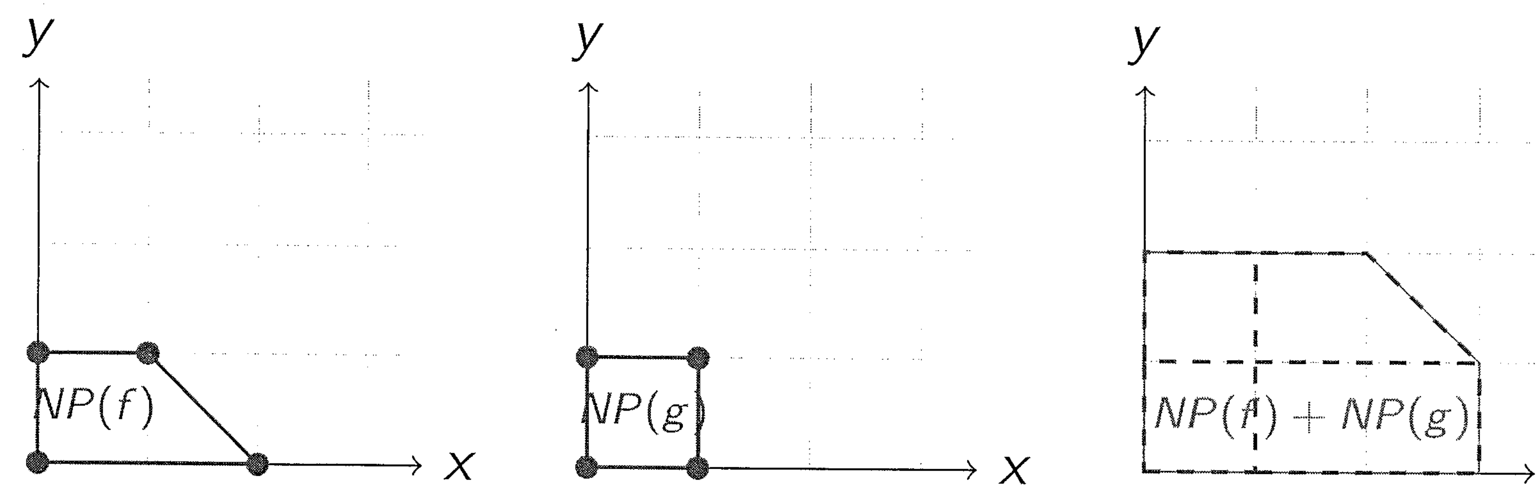
Area($NP(f)$)$= 1.5$

Bernstein's Theorem

What if f and g have *different* Newton polygons?
E.g. $f = x^2 - xy + y - 3$, $g = xy - 2x - 3y - 4$.

Theorem (Bernstein, 1975)

$\#\{f = g = 0\} \leq$ *the* mixed area *of* Newton polygons *of f and g.*



Area($NP(f)$)$= 1.5$ Area($NP(g)$)$= 1$ Area($NP(f) + NP(g)$)$= 5.5$

Mixed Area($NP(f), NP(g)$) $=$ Area($NP(f) + NP(g)$) - Area($NP(f)$) - Area($NP(g)$) $= 3$

Idea: Adjoin the plane by points at infinity cor. to families of weighted curves coming from every *edge* of $NP(f)$ and $NP(g)$.
The new spaces are called *toric varieties*.

Topics related to this talk

- Bezout theorem and weighted Bezout theorem
- Toric varieties, in particular weighted projective spaces
- BKK (Bernstein-Kushnirenko-Khovanskii) theorem
- Roots of polynomials in two variables - Puiseux series