Preliminaries to Chow's Theorem

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Proof of Riemann Extension Thm

From Fact 1, $X = X^r \cup X'$ with $\dim X^r = r$, X' $*$-analytic of dim $< r$.

Take $a \in X^r \backslash X'$. WLOG, assume $a = 0$, $X^r = \mathbb{C}^r \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$ locally.

f analytic on $\{\overline{x}\} \times (\mathbb{C} \backslash \{0\})$, so f extends analytically to $\{\overline{x}\} \times \mathbb{C}$ via:

$$f(\overline{x}, x_n) := \int_{|z|=1} \frac{f(\overline{x}, z)}{z - x_n} dz$$

As $\overline{x}$ varies, $f(\overline{x}, z)$ for fixed $z \neq 0$ varies analytically, so that f extends analytically on $X^r \backslash X'$. Then, by induction on $\dim X$, f extends analytically to U. □

Proof of Mumford's Lemma

Pick an open nbhd U of $f^{-1}(y)$ with $\overline{U}$ compact, and let ∂U be its bdry.

Let (V_n) be decreasing sequence of open nbhds of y with $\overline{V_n}$ compact and $\cap \overline{V_n} = \{y\}$. Then, $\cap_n (f^{-1}(\overline{V_n}) \cap \partial U) = f^{-1}(y) \cap \partial U = \emptyset$. But this is an intersection of countable # of cpcts, so $f^{-1}(\overline{V}) \cap \partial U = \emptyset$ for $V = V_m$.

Let $g : U \cap f^{-1}(V) \to V$ be restriction of f. Then, $\forall\ K \subset U$ compact, $g^{-1}(K) = U \cap f^{-1}(K)$ is compact because $f^{-1}(K)$ is closed, contained in $\overline{U}$ compact and $f^{-1}(K) \cap \partial U \subset f^{-1}(\overline{V}) \cap \partial U = \emptyset$. $\square$

Properties of Resultant

Res is a matrix, and res is a determinant of entries in S, so that if $\exists$ a map $ev_a : S \ni f \mapsto f(a)$, then this commutes with both Res and res .

Claim: $\forall\ P, Q \in \mathbb{C}[w]$, $\text{res}(P, Q) = 0 \Leftrightarrow P$, Q have a common root.

Proof: Denote $p = \deg P$, $q = \deg Q$. $\text{res}(P, Q) = 0 \Rightarrow \exists\ F, G \in \mathbb{C}[w]$ nonzero, $\deg F < q$, $\deg G < p$ with $FP + GQ = 0$, so $FP = -GQ$. Since $\deg F < \deg Q$, not all roots of Q are roots of F, so some roots of Q are roots of P . Conversely, if P , Q have a common root, then trivial. $\square$

Speical Weierstrass Division Thm (as a Fact)

Suppose $c \in \mathbb{C}^d$, $w \in \mathbb{C}$ and $P(a, w) := w^d + \sum_{j=1}^{d} a_j w^{d-j}$. Then, for

$F \in \mathbb{C}\{\overline{x}, w\}$, $\exists\,!\; Q \in \mathbb{C}\{\overline{x}, a, w\}$ and $r_j \in \mathbb{C}\{\overline{x}, a\}$ for $1 \leq j \leq d$

so that $$F(\overline{x}, w) = Q(\overline{x}, a, w)P(c, w) + \sum_{j=1}^{d} r_j(\overline{x}, a) w^{d-j} . \quad (*)$$

Proof of Weierstrass Division Thm using Weierstrass Prep Thm:

$f = uP$ with $u(0) \neq 0$ and $P \in \mathbb{C}\{\overline{x}\}[x_n]$. It suffices to divide g by P via

Special Weierstrass Division Thm. □

Proof of Weierstrass Preparation Thm

Take $F = f$, $w = x_n$. Using Special Weiers. Div. Thm look for solution

of $r_j(\overline{x}, a(\overline{x})) = 0$ with $a(0) = 0 \ \forall\, j$. Then, if $r_j(0,0) = 0 \ \forall\, j$ and

$\det(\frac{\partial r_j}{\partial a_k}(0,0)) \neq 0$, then by Implicit Function Thm, we can find such

solution. Indeed, set $\overline{x} = 0$ and $a = 0$, then by comparing degrees in

$x_n^d(\alpha + \cdots) = Q(0,0,x_n)x_n^d + \sum r_j(0,0)x_n^{d-j} \Rightarrow r_j(0,0) = 0 \ \ \forall\, j$.

Similarly, $Q(0,0,0) = \alpha \neq 0$. Taking $\frac{\partial}{\partial a_k}$ of $(*)$ for each k yields

$0 = Q(0,0,x_n)x_n^{d-k} + \frac{\partial Q}{\partial a_k}(0,0,x_n)x_n^d + \sum_j \frac{\partial r_j}{\partial a_k}(0,0)w^{d-j}$. For $j > k$

by comparing degrees $\Rightarrow \frac{\partial r_j}{\partial c_k}(0,0) = 0$; for $j = k$, $\frac{\partial r_j}{\partial c_j}(0,0) = \alpha \neq 0$.

Summarizing, $(\frac{\partial r_j}{\partial c_k}(0,0))$ is an upper triangular matrix with nonzero

diagonal entries. Hence, $\det(\frac{\partial r_j}{\partial c_k}(0,0)) \neq 0$, and this proves the thm. $\square$