Chow's Theorem

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Definitions: $\mathbb{C}$-analytic and $*$-analytic

Def: Closed set $X \subset U \subset \mathbb{C}^n$ is $\mathbb{C}$-**analytic** if it is locally $V(f_1, \ldots, f_m)$

$:= \{z \in \mathbb{C}^n : f_1(z) = \ldots = f_m(z) = 0\}$ with each f_i locally analytic.

Def: Locally closed $X \subset U$ in $\mathbb{C}^n$ is $*$-**analytic** if $X = \bigcup_{0 \le j \le r} X^j$ with each

X^j $\mathbb{C}$-submanifold of U, $\dim_{\mathbb{C}} X^j = j$, and $\overline{X^j} \subset \bigcup_{0 \le k \le r} X^k$; $\dim_{\mathbb{C}} X := r$.

Remark: If $r < n$, then X is nowhere dense in U, since it is so locally.

Fact 1: $\mathbb{C}$-analytic $\Longrightarrow$ $*$-analytic;

Idea: Express $\mathbb{C}$-analytic X as $\text{Reg}X \sqcup \text{Sing}X$ where $\text{Reg}X$ is r-dim $\mathbb{C}$-manifold and $r = \dim_{\mathbb{C}} X$. It can be done with $\text{Sing}X$ being a $\mathbb{C}$-analytic set. Define $X_1 := \text{Sing}X$. Then, $\dim_{\mathbb{C}} X_1 < r$. Apply above to X_1 . By repeating above, we can get $X = X^r \cup X^{r-1} \cup \cdots \cup X^0$. $\square$

Riemann Extension Theorem: $U \subset \mathbb{C}^n$ open and X analytic in U. Say f is analytic on $U \backslash X$ so that $\forall x \in X$, f is bounded in some nbhd of x. Then, f extends to an analytic function on U.

Other Preliminaries: To be proved in next talk

Weierstrass Preparation Thm: Let $f \in \mathbb{C}\{x\}$, $x := (x_1, \ldots, x_n)$ and

assume $f(0, x_n) = x_n^d(\alpha + x_n g(x_n))$ where $\alpha \in \mathbb{C}^*$, $g \in \mathbb{C}\{x_n\}$ analytic.

Then, $\exists! u \in \mathbb{C}\{x\}$, $u(0) \neq 0$ and $a_i \in \mathbb{C}\{x_1, \ldots, x_{n-1}\} := \mathbb{C}\{\overline{x}\}$,

$1 \leq i \leq d$ with $a_i(0) = 0$ so that $f = u(x_n^d + a_1 x_n^{d-1} + \cdots + a_d)$.

Weierstrass Division Thm: f from above. $\forall g \in \mathbb{C}\{x\}$, $\exists! h \in \mathbb{C}\{x\}$

and $b_i \in \mathbb{C}\{\overline{x}\}$ so that $g = hf + (b_1 x_n^{d-1} + \cdots + b_d)$.

Hilb. Basis Thms. $\mathbb{C}[x]$ and $\mathbb{C}\{x\}$ noetherian rings: ideals are fin. gener.

Mumford's Lemma: Let $f : X \to Y$ be C^0 map of locally compact spaces. Let $y \in Y$ and assume $f^{-1}(y)$ is compact. Then $\exists$ open nbhds $U \subset X$, $V \subset Y$ of $f^{-1}(y)$ such that $f(U) \subset V$ and $f|_U : U \to V$ proper.

Resultant: $\forall P, Q \in S[w]$ for S ring, $\mathrm{res}(P, Q) := \det(\mathrm{Res}(P, Q)) \in S$

where $\begin{array}{ccccc} \mathrm{Res}(P, Q) : \mathrm{Pol}_{\deg Q - 1} & \times & \mathrm{Pol}_{\deg P - 1} & \to & \mathrm{Pol}_{\deg P + \deg Q - 1} \\ (F & , & G) & \mapsto & FP + GQ \end{array}$ and

$\mathrm{Pol}_d \subset S[w]$ with $f \in \mathrm{Pol}_d$ of deg $\leq d$ and a basis of w^j, $j \in \mathbb{N}$.

Evaluation $ev_a : S \ni f \mapsto f(a) \in \mathbb{C}$ commutes with resultant as a map.

For $P, Q \in \mathbb{C}[w]$, $\mathrm{res}(P, Q) = 0 \Leftrightarrow \{w \in \mathbb{C} : P(w) = Q(w) = 0\} \neq \emptyset$.

Chow's Theorem: Projective $\mathbb{C}$-analytic $\Rightarrow$ $\mathbb{C}$-algebraic

as a beautiful **Cor** of **Main Theorem**: $\mathbb{C}$-analytic "$\Longleftarrow$" *-analytic

Proof (Chow): Denote $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ the usual projection.

From Fact 1, given X $\mathbb{C}$-analytic then it is $*$-analytic. Then

$CX := \pi^{-1}(X) \cup \{0\} \subset \mathbb{C}^{n+1}$, is $*$-analytic. Main Theorem implies CX is

$\mathbb{C}$-analytic. Then, near 0 ($\in U$), $CX \cap U$ is $V(f_1, \ldots, f_k)$. and each

$f_i(x) = \sum_r f_{i,r}(x)$, where $f_{i,r}$ is homogeneous polynomial of degree r.

Hence, $\forall |\lambda| \leq 1$, $\forall x^\circ \in CX \cap U$, $0 = f_i(\lambda x^\circ) = \sum_r \lambda^r f_{i,r}(x^\circ)$ and

$\forall f \in \mathbb{C}\{x\} \Rightarrow f(\lambda x^\circ) \in \mathbb{C}\{\lambda\}$. Therefore, $\forall x^\circ \in CX$ all $f_{i,r}(x^\circ) = 0$, and

we can reduce to finite list due to: CX is locally $V(\dots, f_{i,r}, \dots)$.

By Hilb. Basis Thm, it is zero set of finitely many polynomials. □

Now, the proof of the Main Theorem:

Main Thm via Main Step = Projection Lemma

Proof: Induction on $\dim X = r$, $X = X^r \cup X'$, X' $*$-analytic, $\dim X' < r$.

Projection Lemma: $\pi : \mathbb{C}^{n+k} \to \mathbb{C}^n$ projection. $X \subset U \subset \mathbb{C}^{n+k}$ analytic.

$\pi|_X : X \to V \subset \mathbb{C}^n$ proper. Then, $\pi(X) \subset V$ analytic and $\pi|_X$ finite-to-1.

Proof: By induction on k. Factor π into $\mathbb{C}^{n+k} \xrightarrow{\pi_1} \mathbb{C}^{n+k-1} \xrightarrow{\pi_2} \mathbb{C}^n$. Then,

$\pi|_X$ proper $\Rightarrow \pi_1|_X : X \to U_1 := \pi_1(U) \subset \mathbb{C}^{n+k-1}$ proper, so $\pi_1(X) \subset U_1$

analytic, $\pi_1|_X$ finite-to-one. Also $\pi_2|_{\pi_1(X)} : \pi_1(X) \to V$ proper, so the

assertion is true. Suffices to prove base case when $k = 1$.

$\forall y \in V$, $X \cap \pi^{-1}(y) \subset U \cap \pi^{-1}(y) \subset \mathbb{C}$ compact analytic $\Rightarrow X \cap \pi^{-1}(y)$ is finite, so $\pi|_X$ is finite-to-one. Say $\{p_1, \ldots, p_m\} = X \cap \pi^{-1}(y)$. Then, $\exists y \in V_1 \subset V$ open with $X \cap \pi^{-1}(V_1) = \bigcup_{i=1}^{m} U_i$ for $p_i \in U_i$ disjoint open sets, $X_i := X \cap \pi^{-1}(V_1) \cap U_i$ disjoint. Suffices to show $\pi(X_i)$ analytic.

Say $y = 0$, and $X_1 = V(f_1, \ldots, f_p) \in U_1$ with $f_i \in \mathbb{C}\{x_1, \ldots, x_{n+1}\}$. By Weierstrass Preparation and Division Theorem, we may assume

$$f_1 = x_{n+1}^d + \sum_{l=1}^{d} a_l x_{n+1}^{d-l}, \;\; f_i = \sum_{l=1}^{d-1} b_{li} x_{n+1}^{d-l}$$

where $a_l, b_{li} \in \mathbb{C}\{x_1, \ldots, x_n\}, a_l(0) = 0$.

Application of Resultant to Complete the Proof:

With $w := x_{n+1}$, $\operatorname{res}(f_1, t_2 f_2 + \cdots + t_p f_p) = \sum_{|\alpha|=d} t^\alpha R_\alpha$, $R_\alpha \in \mathbb{C}\{x\}$.

Now choose $V_1 \subset V$ small that a_l, b_{li}, R_α converge in V_1, and

$y \in V_1 \Rightarrow$ all roots of $f_1(y, x_{n+1}) = 0$ in U_1. Set $U' := U_1 \cap \pi^{-1}(V_1)$.

Lemma: $\pi(X \cap U') = V(\ldots, R_\alpha, \ldots) \subset V_1$.

Proof: $\forall (z, z_{n+1}) \in X \cap U'$, then $\forall t_i$, $f_1(z, x_{n+1})$, $\sum t_i f_i(z, x_{n+1})$ have

common root $x_{n+1} = z_{n+1}$, so resultant is 0. Thus, $R_\alpha(z) = 0\ \forall \alpha$.

Say, $z^\circ \in V_1$ and $R_\alpha(z^\circ) = 0 \ \forall\alpha$, then $f_1(z^\circ, x_{n+1})$, $\sum t_i f_i(z^\circ, x_{n+1})$

have common root $\forall t$. Say $\alpha_1, \dots, \alpha_d$ roots of $f_1(z^\circ, x_{n+1})$. Call

$W_j := \{t \in \mathbb{C}^{n-1} : \sum t_i f_i(z^\circ, \alpha_j) = 0\}$. Then W_j is a vector space, and

$\cup W_j = \mathbb{C}^{n-1}$. Hence, $W_j = \mathbb{C}^{n-1}$ for some j. Therefore, (z°, α_j) root of

all f_i's, so that it is in $X \cap U'$. ■

This proves the projection lemma with $\pi(X) = V(\dots, R_\alpha, \dots)$ locally. ■

Cor: $\dim \pi(X \cap U') = \dim X$ since π is finite-to-one, cf top of page 15.

Direction (to project along) Lemma and an application

Recall: $X^r \cup X' \subset U$ open, $X' \subset U$ closed analytic, $X^r \subset U \backslash X'$ analytic.

Lemma: $0 \in U \subset \mathbb{C}^n$ open ball, X^r, X' as above, $0 \in X'$, $X^r \cup X' \neq U$.

$\exists$ complex line $l \subset \mathbb{C}^n$ through 0 and a ball $U_1 \ni 0$ in U so that

$X' \cap l \cap U_1 = \{0\}$, $X^r \cap l \cap U_1$ is countable with the only limit point 0 .

Proof: Say $p \in U \backslash (X^r \cup X')$. Say $l = \langle p \rangle$. $p \notin X' \cap l \subset U \cap l$ analytic, so countable, discrete. Set U_1 s.t. $X' \cap l \cap U_1 = \{0\}$. $X^r \cap l \subset (U_1 \cap l) \backslash X'$ analytic, so is countable set possibly accumulating at $X' \cap l$. Shrink U_1 .■

Summarizing Outcome of Main Steps

With $X^r \cup X'$ as above, we project along l from Direction Lemma.

By Mumford's lemma, set $0 \in U_2 \subset U_1 \subset U$ s.th. $\pi : (X^r \cup X') \cap U_2 \to V$

is proper where $0 \in V \subset \mathbb{C}^{n-1}$, $\pi(U_2) \subset V$. By Projection Lemma,

(i) $Y_1 := \pi(X' \cap U_2) \subset V$ analytic, $\pi_{X' \cap U}$ is finite-to-one.

(ii) $Y_0 := \pi((X^r \cap U_2) \backslash \pi^{-1}(Y_1)) \subset V \backslash Y_1$ analytic, π finite-to-1 over Y_0 .

(iii) $\pi((X^r \cup X') \cap U_2) = Y_0 \cup Y_1 \subset V$ similar properties to X^r, X', U .

(iv) $\forall y \in V$, $\pi^{-1}(y)$ is countable.

Locally Biholomorphic Projections $\pi : X^r \cup X' \to V \subset \mathbb{C}^r$

Keep projecting till $\pi : (X^r \cup X') \cap U_2 \to V \subset \mathbb{C}^r$ is proper and surjective.

Proof: π is regular with analytic image (image of of dimension $\leq r$).

If dim $< r$, then $X \cap U_2 =$ countable union of sets of dim $< r$, ?! ■

$\pi(X') = Y_1 \subset V$ analytic, $\pi(X^r \backslash \pi^{-1}(Y_1)) = V \backslash Y_1 \subset V$ open and dense

$X^r \backslash \pi^{-1}(Y_1) \subset X^r$ open and dense. Also, $V \backslash Y_1$ connected, as $Y_1 = \pi(X')$

is analytic of $\dim Y_1 = \dim X' < r = \dim V$ (as π is finite-to-one on X').

$\pi' := \pi|_{X^r \backslash \pi^{-1}(Y_1)} : X^r \backslash \pi^{-1}(Y_1) \to V \backslash Y_1$ proper, finite-to-1 btwn r-mflds.

$\forall z \in X^r \backslash \pi^{-1}(Y_1)$ fixed, say $t_1, \ldots, t_r$ local coordinates around z. Then,

$$J(z) := \frac{\partial(x_1, \ldots, x_r)}{\partial(t_1, \ldots, t_r)}$$

is Jacobian of π on z with x coordinate on U_2 (so $\pi(x) = (x_1, \ldots, x_r)$).

$J(z) \neq 0 \Leftrightarrow \pi$ locally biholomorphic at z. Then, $B_1 \subsetneq X^r \backslash \pi^{-1}(Y_1)$ (where

π is not locally biholomorphic) is locally analytic $V(J) \subsetneq X^r \backslash \pi^{-1}(Y_1)$.

By Projection Lemma, $B := \pi(B_1)$ is analytic in $V \backslash Y_1$.

Since $\dim V(J) < r$, $\dim B < r$, so $V \backslash (Y_1 \cup B)$ connected and dense

in V, so $X^r \backslash \pi^{-1}(B \cup Y_1)$ is dense in $X^r \backslash \pi^{-1}(Y_1)$.

$\pi : X^r \backslash \pi^{-1}(Y_1 \cup B) \to V \backslash (Y_1 \cup B)$ is locally biholomor.

between 2 manifolds of dim $= r$, finite-to-1. This is a covering map. Want to show: $\overline{X^r} \subset \pi^{-1}(V) \subset \mathbb{C}^n$ analytic. Say d is the number of sheets of covering π. Pick linear function L on $\mathbb{C}^n$. Let $V \backslash (Y_1 \cup B) \ni y \mapsto a_i(y)$ be the elementary symmetric function of $L(x_i)$'s with distinct $x_i \in \pi^{-1}(y)$. All $x_i(y)$ are locally analytic $\Rightarrow$ each $a_i(y)$ is analytic. Also, properness of $\pi : \overline{X^r} \to V$ implies a_i's are bounded over $K \backslash (Y_1 \cup B)$ for any compact K **Riemann Extension Theorem** implies a_i extend analytically to V . Then

$F_L(x) := L(x)^d - a_1(\pi(x))L(x)^{d-1} + \cdots + (-1)^d a_d(\pi(x))$ is analytic

on $\pi^{-1}(V)$. Since $F_L = 0$ on $X^r \backslash \pi^{-1}(Y_1 \cup B)$, $F_L = 0$ on $\overline{X^r}$.

Claim: $\overline{X^r} = V(\ldots, F_L, \ldots)$.

Remark $*$. $(n-r-1)d+1$ of L in $(\mathbb{C}^n)^*$ with $L_{|_{\mathbb{C}^r}} = 0$ would do.

Proof: Let $x \in \pi^{-1}(V) \backslash \overline{X^r} \subset \mathbb{C}^n$, let $y = \pi(x)$ and let $y = \lim y_k$,

$y_k \in V \backslash (Y_1 \cup B)$. Then, $\pi^{-1}(y_k) \cap X^r \backslash \pi^{-1}(Y_1 \cup B)$ is $\{x_k^1, \ldots, x_k^d\}$ and

since $\pi : \overline{X^r} \to V$ proper, choose subsequence of (y_k) where $\lim_k x_k^j = x^j$.

Then, $x^j \in \overline{X^r}$, so $x \neq x^j$. Pick L so that $L(x^j) \neq L(x)\ \forall j$ (key for $*$).

End of Proof of Main Theorem

Then, $L(x_k^1), \ldots, L(x_k^d)$ all roots of $t^d - a_1(y_k)t^{d-1} + \cdots + (-1)^d a_d(y_k)$.

$L(x^1), \ldots, L(x^d)$ are the roots of $t^d - a_1(y)t^{d-1} + \cdots + (-1)^d a_d(y)$.

Therefore, $F_L(x) = L(x)^d - a_1(y)L(x)^{d-1} + \cdots + (-1)^d a_d(y) \neq 0$. This shows that $V(\ldots, F_L, \ldots) \subset \overline{X^r}$. The other inclusion is trivial, so that $\overline{X^r} = V(\ldots, F_L, \ldots)$, but we need all $L \in (\mathbb{C}^n)^*$. (Using Remark $*$, we only need $(n-r-1)d+1$ of F's.) But since $\mathbb{C}\{x\}$ is noetherian we can choose $L_1, \ldots, L_u$, so that $\overline{X^r} = V(F_{L_1}, \ldots, F_{L_u})$. □