Poincaré Duality and Harmonic Forms

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MAT 477

March 24, 2014

Differential forms Review

k-forms: $\omega \in \Omega^k(M)$, $\omega(p) \in \Lambda^k(T_pM)^*$, i.e. $\omega(p) : (T_pM)^k \to R$

antisymmetric and linear in each T_pM , e.g. $d\phi(p) : T_pM \ni v \mapsto \frac{\partial \phi}{\partial v}$, or

$(\omega_1 \wedge ... \wedge \omega_k)(v_1, ..., v_k) := \det(\omega_j(v_i))$ for $\omega_j \in (T_pM)^*$, $v_i \in T_pM$.

Smooth $f : M \to N$ induce maps $Df_p : T_p(M) \to T_{f(p)}(N)$, e.g. via

Jacobian Matrices in local coordinates, and $f^* : \Omega^k(N) \to \Omega^k(M)$ via

$(f^*\omega)(p)(v_1, ..., v_k) := \omega(f(p))(Df_p(v_1), ..., Df_p(v_k))$. For $U \subset R^n$ and

n-form $\omega = g dx^1 \wedge ... \wedge dx^n$ let $\int_U \omega := \int_U g dx^1 ... dx^n$. For $f : U \to M$,

Review Continued

$Supp(\omega) \subset f(U)$ we let $\int_M \omega := \int_U f^*\omega$, provided that map f preserves orientation, i.e. linear maps $Df_p : T_p(U) \to T_{f(p)}(M)$ send positive frames into positive frames. For an n-form ω on M let $\int_M \omega := \sum_{i=1}^{k} \int_M \phi_i \omega$.

Def: $\mathrm{O}(M)$:=($\vec{n}(\partial M)$, $\mathrm{O}(\partial M)$), where $\vec{n}(\partial M)$ is the outward normal to smooth boundary ∂M of M , relates orientations of the latter two.

$d : \omega \mapsto d\omega$ is additive with $d(g dx^{i_1} \wedge ... \wedge dx^{i_k}) := dg \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$.

Theorem: (Stokes) For an $(n-1)$-form ω on M holds $\int_M d\omega = \int_{\partial M} \omega$.

Poincaré Duality: $H^k(M), H^{n-k}(M)$ are dual

Theorem: Let M compact, oriented. There exists a non-degenerate pairing $\smile$: $H^k(M) \times H^{n-k}(M) \to \mathbb{R}$ given by $([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta$.

To check this is well-defined, consider $\omega + d\alpha \in [\omega]$, $\eta + d\sigma \in [\eta]$.

$\int_M (\omega + d\alpha) \wedge (\eta + d\sigma) = \int_M \omega \wedge \eta + \omega \wedge d\sigma + d\alpha \wedge \eta + d\alpha \wedge d\eta$.

For any closed form $\omega \in \Omega^k$ and $\alpha \in \Omega^j$ we have that $\omega \wedge d\alpha$ is exact since $d((-1)^k \omega \wedge \alpha) = (-1)^k (d\omega \wedge \alpha + (-1)^k \omega \wedge d\alpha) = \omega \wedge d\alpha$. Thus we find

$\int_M (\omega + d\alpha) \wedge (\eta + d\sigma) = \int_M \omega \wedge \eta + \int_M d\beta = \int_M \omega \wedge \eta + \int_{\partial M} \beta = \int_M \omega \wedge \eta$.

Towards non-degeneracy: The Hodge star

Let $O(M)$ the orientation of M , g a Riemannian metric on M with $\{\partial_i\}$ an orthonormal frame and $\{dx^i\}$ the dual frame positively oriented. For an increasing k-tuple $I = (i_1, ..., i_k)$ use notation $dx^I = dx^{i_1} \wedge ... \wedge dx^{i_k}$.

Definition: The *Hodge star operator* $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ is defined on the basis $\{dx^I\}$ by $\star(dx^{i_1} \wedge ... \wedge dx^{i_k}) = dx^{j_1} \wedge ... \wedge dx^{j_{n-k}}$, where $(j_1, ..., j_{n-k})$ are distinct integers in $\{1, ..., n\} \setminus \{i_1, ..., i_k\}$ ordered so that $[\partial_{i_1}, ..., \partial_{i_k}, \partial_{j_1}, ..., \partial_{j_{n-k}}] = O(M)$, and extended to $\omega \in \Omega^k(M)$ by linearity.

The L^2 Inner Product on $\Omega^k(M)$

Clear that $dx^I \wedge \star dx^I = \text{vol}$ for any increasing k-tuple I , and that

$dx^I \wedge \star dx^J = 0$ for any distinct increasing k-tuples I , J .

Lemma: Inner product $\langle \cdot, \cdot \rangle : \Omega^k(M)^2 \to \mathbb{R}$ given by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta$.

Bilinearity is clear. To check positive definiteness: for $\omega \in \Omega^k(M)$

$\langle \omega, \omega \rangle = \int_M \omega \wedge \star\omega = \int_M (\sum_I \omega_I dx^I) \wedge \star(\sum_J \omega_J dx^J) =$

$\int_M \sum_{I,J} \omega_I \omega_J dx^I \wedge \star dx^J = \int_M (\sum_I \omega_I^2)\text{vol} \ \geq 0$ since $(\sum_I \omega_I^2) \geq 0$.

Equality holds if and only if $(\sum_I \omega_I^2) \equiv 0$ if and only if $\omega \equiv 0$. ■

Proof of non-degeneracy

Lemma: $\forall$ cohomology class $[\tilde{\omega}] \in H^k(M)$, $\exists\, \omega \in [\tilde{\omega}]$ s.th. $d \star \omega = 0$.

Now, let $[\tilde{\omega}] \in H^k(M)$, and $\omega \in [\tilde{\omega}]$ as above. Since $d \star \omega = 0$, we have that $[\star\omega] \in H^{n-k}(M)$ makes sense. Further, $[\tilde{\omega}] \smile [\star\omega] = \int_M \omega \wedge \star\omega \geq 0$ with equality if and only if $\omega \equiv 0$, if and only if $[\tilde{\omega}] = 0$, done. ■

Forms of the type above, i.e. such that $d\omega = d \star \omega = 0$, are called harmonic. We will now prove a generalization of the above lemma:

Theorem: $\exists$ a unique harmonic representative of each cohomology class.

Thm: $\exists$ unique harmonic rep. of each cohomology class.

Given the inner product, we construct a formal adjoint d^* to d :

Proposition: There is $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ with $\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle$

for $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$. Claim $d^* = (-1)^{n(k+1)+1} \star d\star$ will do:

$d(\alpha\wedge\star\beta) = d\alpha\wedge\star\beta + (-1)^{k-1}\alpha\wedge d\star\beta = d\alpha\wedge\star\beta + (-1)^{n(k+1)+2}\alpha\wedge\star\star d\star\beta$.

Noting $\star^2 = (-1)^{k(n-k)}\text{id}$ on $\Omega^k(M)$. Integrating both sides, we obtain

$0 = \int_M d(\alpha \wedge \star\beta) = \int_M d\alpha \wedge \beta - \int_M \alpha \wedge \star((-1)^{n(k+1)+1} \star d \star \beta)$.

The first equality follows from Stokes' theorem. This is the result.

The Hodge Laplacian Δ on $\Omega^k(M)$

Definition: $\Delta := d^*d + dd^* : \Omega^k(M) \to \Omega^k(M)$. For $f \in \Omega^0(M)$

$\Delta f = \star d \star (\sum_i \partial_i f \ dx^i) = \pm \star d(\sum_i \partial_i f \ (-1)^i dx^1 \wedge ... \wedge \hat{d}x^i \wedge ... \wedge dx_n) =$

$\pm \star (\sum_i \partial^2_{ii} f \ \text{vol}) = -\sum_i \partial^2_{ii} f$ agreeing with the usual Δ .

Proposition: $\ker \Delta = \ker d \cap \ker d^*$. Clearly $\ker d \cap \ker d^* \subset \ker \Delta$.

Also $\langle \Delta\omega, \omega \rangle = \langle d^*d\omega, \omega \rangle + \langle dd^*\omega, \omega \rangle = |d\omega|^2 + |d^*\omega|^2$ so that if

$\Delta\omega = 0$ then $|d\omega|^2 = |d^*\omega|^2 = 0$ implying $\omega \in \ker d \cap \ker d^*$, as claimed.

Definition: $\omega \in \Omega^k(M)$ is called harmonic if $\omega \in ker\Delta$.

Harmonic Representatives of Cohomology Classes exist

Let $\mathcal{H}_j$ be Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ a continuous linear operator.

Then $\ker T := T^{-1}(0)$ is a closed subspace, thus defines a Hilbert space.

Further, $(\text{im } T)^{\perp} = \ker T^*$ since $w \in (ImT)^{\perp}$ iff $\langle Tv, w \rangle = 0 \ \forall \ v \in \mathcal{H}_1$

iff $\langle v, T^* w \rangle = 0 \ \forall \ v \in \mathcal{H}_1$ iff $w \in \ker T^*$. In sum, if im T closed then

$\mathcal{H} = \text{im } T \oplus \ker T^*$. Apply this (only formally!) to d on L^2 forms.

First, $\ker d^k$ will be a Hilbert space. Then since d^{k-1} maps into $\ker d^k$ we

should be able to decompose $\ker d^k = \text{im } d^{k-1} \oplus \ker(d^*|_{\ker d^k})$.

Harmonic Representatives of Cohomology Classes

$\ker(d^*|_{\ker d^k}) = \ker d^* \cap \ker d = \ker \Delta$. So, $\ker d^k = \text{im } d^{k-1} \oplus \ker \Delta$.

<u>Fact:</u> (hard analysis) $\ker \Delta \subset \Omega^k(M)$; harmonic forms are C^∞ .

All of the above work was sketched thinking of the operators as acting on (a dense subset of) L^2 forms. Using this fact, we can intersect both sides of the first line with $\Omega^k(M)$ to find the same decomposition for $\ker d^k|_{\Omega^k(M)}$. Thus we have: $H^k(M) = \ker d^k / \text{im } d^{k-1} \cong \ker \Delta$ and so obtain a harmonic representative of each cohomology class.