De Rham Theorem à la Whitney. Part 1.

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Differential forms on C^∞ compact n-dim manifolds M :

PofU: $\exists\ \phi_i \in C^\infty(M)$, $\sum_i \phi_i(x) \equiv 1$, $Supp(\phi_i) \subset K_i \Subset V_i \xrightarrow{f_i} R^n$

(V_i coord. nbhds), e.g. $f_i(K_i) = \{|y| \le 1\}$, $M = \bigcup_i f_i^{-1}(\{|y| < 1\})$

then $\psi_i(x) := \exp(1/(|f_i(x)|^2 - 1))$ and $\phi_i := \psi_i / \sum_i \psi_i$ will do.

k-forms: $\omega \in \Omega^k(M)$, $\omega(p) \in \Lambda^k(T_pM)^*$, i.e. $\omega(p) : (T_pM)^k \to R$

antisymmetric and linear in each T_pM , e.g. $d\phi(p) : T_pM \ni v \mapsto \frac{\partial \phi}{\partial v}(p)$,

or $(\omega_1 \wedge ... \wedge \omega_k)(v_1, ..., v_k) := \det(\omega_j(v_i))$ for $\omega_j \in (T_pM)^*$, $v_i \in T_pM$

Smooth $f : M \to N$ induce maps $Df_p : T_p(M) \to T_{f(p)}(N)$, e.g. via

Jacobian Matrices in local coordinates, and $f^* : \Omega^k(N) \to \Omega^k(M)$ via

$(f^*\omega)(p)(v_1, ..., v_k) := \omega(f(p))(Df_p(v_1), ..., Df_p(v_k))$. For $U \subset R^n$ and

n-form $\omega = g dx_1 \wedge ... \wedge dx_n$ let $\int_U \omega := \int_U g dx_1 ... dx_n$. For $f : U \to M$ and

$Supp(\omega) \subset f(U)$ we let $\int_M \omega := \int_U f^*\omega$, provided that map f preserves

orientation, i.e. linear maps $Df_p : T_p(U) \to T_{f(p)}(M)$ send positive frames

into positive frames. For an n-form ω on M let $\int_M \omega := \sum_{i=1}^k \int_M \phi_i \omega$.

Def: O(M):=($\vec{n}(\partial M)$, O(∂M)), where $\vec{n}(\partial M)$ is the outward normal

to smooth boundary ∂M of M , relates orientations of the latter two.

Stoke's Thm for oriented M with boundary ∂M smooth.

Theorem: For an $(n-1)$-form ω on M holds $\int_M d\omega = \int_{\partial M} \omega$, where

$d : \omega \mapsto d\omega$ is additive with $d(g dx_{i_1} \wedge \ldots \wedge dx_{i_k}) := dg \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

Ex: $\int_{[a,b]} df := \int_a^b f' dt = f(b) - f(a) =: \int_{\partial[a,b]} f$ (Fund. Thm of Calc.)

Conventions: V coord. charts, if $V \cap \partial M \neq \emptyset$ we choose coord. so that

$\partial M = \{x_1 = 0\}$; set $dx_1 \wedge .\hat{j}. \wedge dx_n := dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_n$

Proof: WLOG assume that our coordinate charts V are nbhds of cubes

$Q := \{x_j^0 \leq x_j \leq x_j^1\}_{1 \leq j \leq n}$ with $x_1^i = i$ and interiors 'covering' $M \setminus \partial M$;

that $\partial M \cup \mathrm{int} Q \supset Supp(\omega)$ and $\omega := g(x) dx_1 \wedge ..\hat{j}. \wedge dx_n$, with "int" short for interior. Then $g_{|_{x_j = x_j^i}} = 0 \ \forall \ (j, i) \neq (1, 0)$, where $i = 1$ or 0 , and $d\omega = (-1)^{j-1} \frac{\partial g}{\partial x_j} dx_1 \wedge ... \wedge dx_n$. Let $K := Supp(\omega)$.

Case 1: $j \neq 1$ or $K \subset \mathrm{int} Q$. Then $\int_{\partial Q} \omega = 0$ and $\int_Q d\omega =$

$\int_Q (-1)^{j-1} \frac{\partial g}{\partial x_j} dx_1 ... dx_n = (-1)^{j-1} \int_Q \frac{\partial g}{\partial x_j} dx_1 ... dx_n = 0$ use FundThmCalc.

Case 2: $j = 1$, $K \cap \partial M \neq \emptyset \ \Rightarrow \omega_{|\partial Q} = g(x)_{|_{x_1 = 0}} dx_2 \wedge ... \wedge dx_n$ and

$\int_Q d\omega = \int_Q \frac{\partial g}{\partial x_1} dx_1 ... dx_n = - \int_{\{x_1 = 0\}} g(x) dx_2 ... dx_n = \int_{\partial Q} \omega$. ■

Poincare Lemma for Contractible Manifolds.

Poincare Lemma: Assume M is contractible, i.e. $\exists\ p_0 \in M$ and map

$H : MxR \to M$ s.th. $H(p,0) = p_0$, $H(p,1) = p\ \forall p \in M$; $\omega \in \Omega^k(M)$.

Then $d\omega = 0$ iff $\exists\ \beta$ s.th. $d\beta = \omega$ (in words: ω is closed iff it is exact).

Proof: With $\pi : MxR \ni (x,t) \mapsto x \in M$, $\omega \in \Omega^k(M)$ set $\bar{\omega} := H^*\omega$

$= \omega_1 + dt \wedge \eta$; ω_1 , η having no dt $\Rightarrow \omega_{1_{|_{t=0}}} = 0$ and $\omega_{1|t=1} = \omega$.

For $\{\vec{v}_i\}_{1\leq i\leq k-1} \subset T_pM$ let $g(p,t) := \eta(p,t)(\vec{v}_1 , ..., \vec{v}_{k-1})$

and $I : \Omega^k(MxR) \to \Omega^{k-1}(M)$ s.th. $(I\bar{\omega})(\vec{v}_1 , ..., \vec{v}_{k-1}) := \int_0^1 g(p,t)dt$.

Sublemma: $\bar{\omega}_{|_{t=1}} - \bar{\omega}_{|_{t=0}} = d(I\bar{\omega}) + I(d\bar{\omega})$. **Proof:** Suffices to show

Case 1: $\omega_1 = f dx_{i_1} \wedge ... \wedge dx_{i_k} \Rightarrow d\omega_1 = \frac{\partial f}{\partial t} dt \wedge dx_{i_1} ... \wedge dx_{i_k} + d_x \omega_1$,

$I(\omega_1) = 0$ and $I(d\omega_1) = (\int_0^1 \frac{\partial f}{\partial t} dt) dx_{i_1} \wedge ... \wedge dx_{i_k} = (\omega_1)|_{|_{t=1}} - (\omega_1)|_{|_{t=0}}$.

Case 2: $\eta = f dx_{i_1} \wedge ... \wedge dx_{i_{k-1}} \quad \Rightarrow \quad -I(d(dt \wedge \eta)) = I(dt \wedge d\eta) =$

$\sum_\alpha (\int_0^1 \frac{\partial f}{\partial x_\alpha} dt) dx_\alpha \wedge dx_{i_1} \wedge ... \wedge dx_{i_{k-1}} = d[(\int_0^1 f dt) dx_{i_1} \wedge ... \wedge dx_{i_{k-1}}]$

$= d(I(dt \wedge \eta))$, which completes the proof.

Since $d\omega = 0 \Rightarrow d\bar{\omega} = 0$ and due to $\bar{\omega}_{|_{t=1}} = \omega$ and $\bar{\omega}_{|_{t=0}} = 0$

Poincare Lemma with $\beta := I\bar{\omega}$ follows from our sublemma. ■

Simplices, Complexes and Triangulation of Manifolds M .

Simplex $\sigma = v_0 ... v_m$, $v_i \in R^n$ is $\{x : x = \sum b_i v_i, b_i \geq 0, \sum b_i = 1\}$.

Face of σ is a simplex spanned by a subset of vertices of σ . Complex K is

a collection of simplices s.th. 1) $\sigma \in K \Rightarrow$ faces of σ are in K ;

2) $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$ is face of σ and τ . Ordering vertices orients σ .

Fact: Manifolds admit triangulation $T(M)$, i.e. $\exists$ homeo. $\pi : K \stackrel{onto}{\rightarrow} M$

s.th. $\forall\, \sigma \in K\, \exists$ coord. nbhd. $V \supset \pi(\sigma)$ s.th. $f^{-1} \circ \pi$ is affine in σ .

$\Sigma_r :=$ R-vec. space of r-chains $A = \sum_i a_i \sigma_i^r$; cochains $\in$ $\Sigma^r := \Sigma_r^{dual}$.

Boundaries, Coboundaries, Chain Complexes.

$\partial_{r-1} \sum_i a_i \sigma_i^r := \sum_i a_i \partial_{r-1} \sigma_i^r = \sum_i a_i \sum_{j=0}^r (-1)^j (v_0 ... \hat{v}_j ... v_r)$ for $\sigma_i^r :=$

$(v_0 ... v_r)$, defines boundaries map $\partial_{r-1} : \Sigma_r \to \Sigma_{r-1}$ and $\partial_{r-1} \circ \partial_r = 0$.

Coboundaries map $\partial^*_{r-1} : \Sigma^{r-1} \to \Sigma^r$ is dual to ∂_{r-1} . Finally, due to the

Stokes Thm $Int^r(\omega)(\sum_i a_i \sigma_i^r) := \sum_i a_i \int_{\sigma_i^r} \omega$ defines 'homomorphism' of

complexes $\Omega^r(M) \overset{Int^r}{\to} \Sigma^r$, i.e. the following diagram is commutative:

$$
\begin{array}{ccccccc}
... \to & \Omega^{k-1}(M) & \overset{d_{k-1}}{\to} & \Omega^k(M) & \overset{d_k}{\to} & \Omega^{k+1}(M) & \to ... \\
& \downarrow Int^{k-1} & & \downarrow Int^k & & \downarrow Int^{k+1} & \\
... \to & \Sigma^{k-1} & \overset{\partial^*_{k-1}}{\to} & \Sigma^k & \overset{\partial^*_k}{\to} & \Sigma^{k+1} & \to ...
\end{array}
$$

Elementary Forms provide a right inverse to Int^k .

Definition: $St(\sigma) = \bigcup_{\sigma \in F(\tau)} int(\tau)$, $F(\tau) =$ faces of τ .

$O := \{St(q_i)\}$ is open cover of M ; $\exists$ part. of unity $\{\phi_i\}$ subord. to O .

Let $[\sigma] \in \Sigma^k$ be the dual basis to the one formed by simpleces $\sigma \in \Sigma_k$.

Thm: $\Phi^k[q_{\lambda_0}, ..., q_{\lambda_k}] := k! \sum_{i=0}^{k} (-1)^i \phi_{\lambda_i} d\phi_{\lambda_0} \wedge ...\hat{i}... \wedge d\phi_{\lambda_k} \in \Omega^k(M)$

extended to Σ^k as linear satisfies: 1) $supp(\Phi^k[\sigma]) \subset St(\sigma)$;

2)$\Phi^k \partial^* X = d\Phi^{k-1} X$; 3)$Int^k \circ \Phi^k X = X$; 4)$\Phi^0 I^0 = \mathbf{1}$, $I^0 := \sum_r [q_r]$.

Cor: $\Omega^\bullet = ker(Int^\bullet) \bigoplus \Phi^k(\Sigma^\bullet)$ and $\Phi^k : \Sigma^\bullet \to \Phi^k(\Sigma^\bullet)$ is an isomorphism.

Proof: To begin with $Supp(\phi_i) \subset St(q_i)$ implies 1) and

$\Phi^0 I^0 = \Phi^0(\sum_r [q_r]) = \sum_r (\Phi^0([q_r])) = \sum_r \phi_r = \mathbf{1}$ implies 4) .

Suffices to show 2) for $X := [\sigma] = [q_{\lambda_0} ... q_{\lambda_k}]$. We'll use:

(i) $\partial^* X = \sum_r^* [q_r \sigma]$, where the sum $\sum_r^*$ is over q_r s.th. $(q_r \sigma) \in \Sigma_{k+1}$;

(ii) $\forall$ q_r in ** , i.e. $(q_r \sigma) \notin \Sigma_{k+1}$, holds $\cap_{0 \leq i \leq k} St(q_{\lambda_i}) \cap St(q_r) = \emptyset$

$\Rightarrow$ $\cap_{0 \leq i \leq k} Supp \phi_{\lambda_i} \cap Supp \phi_{q_r} = \emptyset$. Both proved in the Appendix.

Of course $\sum_{i=0}^k d\phi_{\lambda_i} + \sum_{r \neq \lambda_i \ \forall i} d\phi_r = 0$, since $d(\sum \phi_i) = 0$, and

$d\Phi^k [q_{\lambda_0} ... q_{\lambda_k}] = (k+1)! d\phi_{\lambda_0} \wedge ... \wedge d\phi_{\lambda_k}$. Using all that, we show:

Elementary Forms: proof of property 2) .

$\frac{1}{(k+1)!}\Phi^{k+1}\partial^*[q_{\lambda_0}...q_{\lambda_k}] = \frac{1}{(k+1)!}\sum_r^* \Phi^{k+1}[q_r q_{\lambda_0}...q_{\lambda_k}] =$

$\sum_r^*[\phi_{q_r} d\phi_{\lambda_0} \wedge ... \wedge d\phi_{\lambda_k} - \sum_{i=0}^k (-1)^i \phi_{\lambda_i} d\phi_{q_r} \wedge d\phi_{\lambda_0} \wedge ...\hat{i}... \wedge d\phi_{\lambda_k}] =$

$\sum_r^* \phi_{q_r} d\phi_{\lambda_0} \wedge ... \wedge d\phi_{\lambda_k} + \sum_r^{**} d\phi_{q_r} \wedge \sum_{i=0}^k (-1)^i \phi_{\lambda_i} d\phi_{\lambda_0} \wedge ...\hat{i}... \wedge d\phi_{\lambda_k} +$

$\sum_{j=0}^k d\phi_{\lambda_j} \wedge \sum_{i=0}^k (-1)^i \phi_{\lambda_i} d\phi_{\lambda_0} \wedge ...\hat{i}... \wedge d\phi_{\lambda_k}$

$= (\sum_r^* + \sum_r^{**})\phi_{q_r} d\phi_{\lambda_0} \wedge ... \wedge d\phi_{\lambda_k} + \sum_{i=0}^k \phi_{\lambda_i} d\phi_{\lambda_0} \wedge ... \wedge d\phi_{\lambda_k} =$

$d\phi_{\lambda_0} \wedge ... \wedge d\phi_{\lambda_k} = \frac{1}{(k+1)!} d\Phi^k[q_{\lambda_0}...q_{\lambda_k}]$, which proves 2) .

Proof of 3) : Map Φ^k as a right Inverse to Int^k .

Induction on k : $k = 0$, $Int^0(\Phi^0[q_i]) \cdot q_j = (\Phi^0[q_i])(q_j) = \phi_i(q_j) = \delta_{ij}$

(for $i \neq j$ since $q_j \notin St(q_i) \Rightarrow \phi_i(q_j) = 0$, and using $\sum \phi_i(q_j) = 1$ for

$i = j$). When $k > 0$: $\sigma \neq \tau \Rightarrow \tau \subset M \setminus St(\sigma) \Rightarrow Int^k(\Phi^k[\sigma]) \cdot \tau = 0$

using 1) . When $\sigma = \tau$ let $\partial\sigma = \alpha +$[other $(k-1)$-faces of σ]. Then

$\int_\sigma \Phi^k[\sigma] = \int_\sigma \Phi^k \partial^*[\alpha] = \int_\sigma d\Phi^{k-1}[\alpha] = \int_{\partial\sigma} \Phi^{k-1}[\alpha] = \int_\alpha \Phi^{k-1}[\alpha] = 1$

with the last equality due to the inductive assumption, as required.

Appendix: Proof of $\partial^*[\sigma] = \sum_r^*[q_r\sigma]$.

It sufficies to show that $\partial^*[\sigma](A) := [\sigma](\partial A) = \sum_r^*[q_r\sigma](A)$ for any

$A \in \Sigma_{k+1}$. Of course if σ is not one of the faces of A then

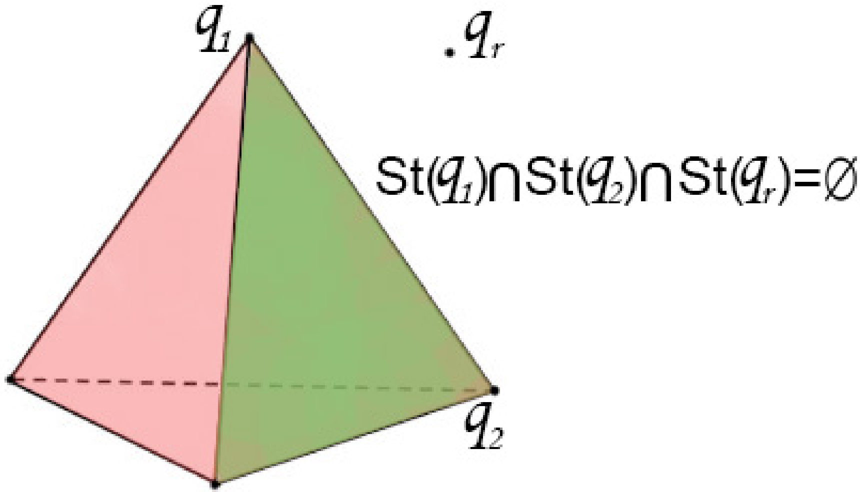
$[\sigma](\partial A) = 0$ and $\sum_r^*[q_r\sigma](A) = 0$ since then simplex $A \neq q_r\sigma$.

Therefore, it suffices to consider $A = A_s := q_s\sigma \in \Sigma_{k+1}$.

Then for all $A_s \in \Sigma_{k+1}$ holds $[\sigma](\partial A_s) = 1$ and $\sum_r^*[q_r\sigma](A_s) = 1$.

Therefore $\partial^*[\sigma] = \sum_r^*[q_r\sigma]$, as required.

Appendix: Claim (ii) from 2) of page 11.



Appendix: Proof of (ii) from 2) of page 11.

Assume $(q_r q_{\lambda_o}...q_{\lambda_k}) \notin \Sigma_{k+1}$. If $Z := \bigcap_{0\leq i\leq k} St(q_{\lambda_i}) \cap St(q_r) \neq \emptyset$

then set Z consists of a union of simpleces (as each $St(q)$ is) and,

moreover, simplex $\tau \subset Z$ iff its vertices include all q_r , q_{λ_o} , ..., q_{λ_k} !

But these vertices then span face $(q_r q_{\lambda_o}...q_{\lambda_k})$ of τ . By definition of

the complex of triangulation then $(q_r q_{\lambda_o}...q_{\lambda_k}) \in \Sigma_{k+1}$ contrary to our

assumption, i.e. $Z = \emptyset$ as claimed in (ii) . ■

Denote: $\chi(T(M)) := \sum_{k=0}^{n}(-1)^k \#\{\sigma \in T(M) : dim\ \sigma = k\}$.

Invariance of $\chi(T(M))$ on triangulation $T(M)$ of M .

Fact: $ker(Int^\bullet)$ is acyclic subcomplex of $\Omega^\bullet$, i.e. $ker(d_{k|ker(Int^k)}) =$

$Im(d_{k-1|ker(Int^{k-1})})$, $\Rightarrow$ via **Cor** (from p.10) that $\frac{ker(\partial_k^*)}{Im(\partial_{k-1}^*)} \cong \frac{ker(d_k)}{Im(d_{k-1})}$.

Note: $\#\{\sigma \in T(M) : dim\ \sigma = k\} = dim_R \Sigma_k = dim_R \Sigma^k$.

Thm: Euler characteristic $\chi(M) := \sum_{k=0}^n (-1)^k dim_R \frac{ker(d_k)}{Im(d_{k-1})} = \chi(T(M))$

Corollary: $\chi(T(M))$ does not depend on triangulation $T(M)$ of manifold.

Indeed, $dim_R \Sigma^k = dim_R Im(\partial_k^*) + dim_R \frac{ker(\partial_k^*)}{Im(\partial_{k-1}^*)} + dim_R Im(\partial_{k-1}^*)$.

Hence $\chi(M) = \sum_{k=0}^n (-1)^k dim_R \frac{ker(\partial_k^*)}{Im(\partial_{k-1}^*)} = \chi(T(M))$. ■

Triangulation of 2-handles:

$\mathbf{v.} = 18$, $\mathbf{e.} = 60$, $\mathbf{t.} = 40 \;\Rightarrow\; \chi(M) = -2$.