Favourite Theorems of the last year. Students choice (notes unaltered).

Class MAT477 in 2013 - 2014

April, 2014

Changho Han.
Normalization Theorem (Preliminaries).

Def: Projective Plane Curve is of the form $V(F) \subset \mathbb{CP}^2$ where $F \neq 0$ is a homogeneous polynomial in 3 variables.

Def: Projective Plane Curve X is irreducible if $\exists F$ irreducible as a polynomial and $X = V(F)$.

Def: $x \in X$ projective plane curve is singular when X is not smooth at x.

Def: Riemann surface X is a connected 1-dimensional complex manifold.

Fact: Any projective plane curve X has only finitely many singular points.

Normalization Theorem (Statement)

Def: $S_X := \{x \in X : x \text{ singular in } X\}$ for projective plane curve X.

Def: Normalization of projective plane curve X is (Y, σ) where Y is a

compact Riemann surface, $\sigma : Y \to X$ surjective holomorphic map,

$\sigma^{-1}(S_X)$ is finite, and $\sigma|_{Y \setminus \sigma^{-1}(S_X)} : Y \setminus \sigma^{-1}(S_X) \to X \setminus S_X$ is bijective.

Normalization Theorem: Given projective plane curve X, there exists

normalization (Y, σ) unique upto biholomorphic maps.

RongXi Guo
Bezout Theorem in dimension 2 .

Def: Given field F and point $P \in F^2$, the ring of rational functions $\frac{f}{g}$,

$(f,\ g \in F[x,y], g(P) \neq 0)$ is called the local ring at P, denoted by O_P.

Let A , B be two plane curves with corresponding functions

$f(x,y) = 0$, $g(x,y) = 0$ where $f,\ g \in C[x,y]$.

Def: Let $(f,g)_P$ be the ideal $O_P f + O_P g$ in O_P generated by f and g,

the intersection multiplicity at point P of curves A and B rmis

$I_P(A,B) \equiv I_P(f,g) \equiv dim_C O_P/(f,g)_P$.

Bezout Theorem:

If f, g have no common factor, A , B intersect at precisely mn points,

counting all multiplicities and intersections at infinity (short as iai.) .

i.e. $\sum_{P \in C^2} I_P(f,g) + N_{inf} = mn$, where N_{inf} is the number of iai.,

i.e. the extra intersections in the extended projective space .

When f, g have common factor(s)

$\Rightarrow$ All points of common factor(s) are on the curves of both A and B

$\Rightarrow$ A, B have infinite common points .

Alex Edmonds.
Banach-Tarski Paradox: Beautiful or Disturbing?

Main Thm: For $B \subset \mathbb{R}^n$ open or closed ball, $\exists\ S_1, \ldots, S_5 \subset B$ disjoint,

$\exists\ \varphi_1 \ldots \varphi_5$ isometries of $\mathbb{R}^n$ s.th.

$$B = S_1 \cup \cdots \cup S_5 = \varphi_1(S_1) \cup \varphi_2(S_2) = \varphi_3(S_3) \cup \varphi_4(S_4) \cup \varphi_5(S_5)$$

Axiom of Choice (AC): Proof uses AC. Cited as reason not to accept AC.

Reconciling Intuition: Intuition says mass of whole should be sum of mass of parts. This logic fails since $S_1, \ldots, S_5$ are non-measurable.

Converse (Fact): To construct non-measurable sets requires AC.

Aaron Crighton.
Ultraproducts and Los' Theorem.

Introduction. Preliminaries: With $\mathscr{P}(S) := \{U : U \subset S\}$,

Def 1: Given a set S, an **ultrafilter** is a subset $\mathscr{U} \subset \mathscr{P}(S)$ s.th:

i) $\emptyset \notin \mathscr{U}$; ii) $X_1, X_2 \in \mathscr{U} \Rightarrow X_1 \cap X_2 \in \mathscr{U}$;

iii) $X_1 \subset X_2$ and $X_1 \in \mathscr{U} \Rightarrow X_2 \in \mathscr{U}$; iv) $X \notin \mathscr{U} \Rightarrow X^{\mathrm{c}} \in \mathscr{U}$.

Def 2: Given sets $\langle S_\beta \rangle_{\beta \in I}$ indexed by a set I, and an ultrafilter $\mathscr{U}$ on I,

the reduced product of $\langle S_\beta \rangle_{\beta \in I}$ is defined as the quotient $(\prod_{\beta \in I} S_\beta)/\sim$

where $(\prod_{\beta \in I} S_\beta)$ is cartesian prodcut and $\sim$ is the equivalence relation:

$f \sim g \iff \{\beta : f(\beta) = g(\beta)\} \in \mathscr{U}$. We denote this set by $\prod_{\mathscr{U}} S_\beta$.

Def 3: Given $\mathbb{L}$ a 1^{st}-order language with relation symbols R_α, constants

c_α and functions f_α and a class of models $\langle \mathscr{M}_\beta \rangle_{\beta \in I}$ for $\mathbb{L}$ indexed by set I,

and ultrafilter $\mathscr{U}$ on I, the **ultraproduct** of $\langle \mathscr{M}_\beta \rangle_{\beta \in I}$, denoted by $\prod_{\mathscr{U}} \mathscr{M}_\beta$

is the model for the language $\mathbb{L}$ defined as:

U.i) $|\prod_{\mathscr{U}} \mathscr{M}_\beta| = \prod_{\mathscr{U}} |\mathscr{M}_\beta|$

U.ii) $R_\alpha^{\prod_{\mathscr{U}} \mathscr{M}_\beta} = \prod_{\mathscr{U}} R_\alpha^{\mathscr{M}}$

Definition of Ultraproduct continued

U.iii)$c_\alpha^{\prod_{\mathscr{U}} \mathscr{M}_\beta} = [(c_\alpha^{\mathscr{M}_\beta})_{\beta \in I}]$ (the equivalence class of the element

$(c_\alpha^{\mathscr{M}_\beta})_{\beta \in I} \in \prod_{\beta \in I} \mathscr{M}_\beta$ mod the relation $\sim$)

U.iv) If F_α is an n-ary function then $f_\alpha^{\prod_{\mathscr{U}} \mathscr{M}_\beta}$ is the function from

$(\prod_{\mathscr{U}} |\mathscr{M}_\beta|)^n \to \prod_{\mathscr{U}} |\mathscr{M}_\beta|$ defined (with $g := (g_1, \ldots g_n), g_i \in \prod_{\mathscr{U}} |\mathscr{M}_\beta|$):

$f_\alpha^{\prod_{\mathscr{U}} \mathscr{M}_\beta}(g) = [(f_\alpha^{\mathscr{M}_\beta}(g(\beta)))_{\beta \in I}]$ (Exercise: this is well-defined)

Los' Theorem: Given a formula $\Phi(v)$ $(v := (v_1, \ldots, v_n))$ in

language $\mathbb{L}$, ultraproduct $\prod_{\mathscr{U}} \mathscr{M}_\beta$ and $g := (g_1, \ldots, g_n)$ $(g_i \in |\prod_{\mathscr{U}} \mathscr{M}_\beta|)$

then, $\prod_{\mathscr{U}} \mathscr{M}_\beta \models \Phi(v)[g] \iff \{\beta : \mathscr{M}_\beta \models \Phi(v)[g(\beta)]\} \in \mathscr{U}$

Proof: By induction on complexity of Φ Write $\mathscr{M} := \prod_{\mathscr{U}} \mathscr{M}_\beta$:

Case 1: $\Phi(v)$ is of the form $t_1(v) = t_2(v)$ for terms t_1, t_2 in $\mathbb{L}$. Then,

$\mathscr{M} \models \Phi(v)[g] \iff t_1^{\mathscr{M}}(g) = t_2^{\mathscr{M}}(g) \iff$

$\{\beta : t_1^{\mathscr{M}_\beta}(g(\beta)) = t_2^{\mathscr{M}_\beta}(g(\beta))\} \in \mathscr{U} \iff \{\beta : \mathscr{M}_\beta \models \Phi(v)[g(\beta)]\} \in \mathscr{U}$

Case 2: Φ is of the form $\Theta \wedge \Psi$ where the result holds for Ψ and Θ

$\mathscr{M} \models \Phi(v)[g] \iff \mathscr{M} \models \Psi(v)[g]$ and $\mathscr{M} \models \Theta(v)[g]$

$\iff \{\beta : \mathscr{M}_\beta \models \Psi(v)[g(\beta)]\} \in \mathscr{U}$ and $\{\beta : \mathscr{M}_\beta \models \Theta(v)[g(\beta)]\} \in \mathscr{U}$

And by properties ii) and iii) of ultrafilters,

$\iff \{\beta : \mathscr{M}_\beta \models \Psi(v)[g(\beta)]\} \cap \{\beta : \mathscr{M}_\beta \models \Theta(v)[g(\beta)]\}$

$= \{\beta : \mathscr{M}_\beta \models (\Theta \wedge \Psi)(v)[g(\beta)]\} = \{\beta : \mathscr{M}_\beta \models \Phi(v)[g(\beta)]\} \in \mathscr{U}$

Case 3: Φ is of the form $\neg\Psi$ where the result holds for Ψ. Then,

$\mathscr{M} \models \Phi(v)[g] \iff$ It is not the case that $\mathscr{M} \models \Psi(v)[g] \iff$

$\{\beta : \mathscr{M}_\beta \models \Psi(v)[g(\beta)]\} \notin \mathscr{U} \overset{byU.iv)}{\iff} \{\beta : \mathscr{M}_\beta \models \Phi(v)[g(\beta)]\} \in \mathscr{U}$.

Case 4: Φ is of the form $(\exists x)\Psi(v,x)$ and result holds for $\Psi(v,x)$. Then,

$\mathscr{M} \models \Phi(v)[g] \Longrightarrow \mathscr{M} \models \Psi(v,x)[g,h]$ for some $h \in |\mathscr{M}| \Longrightarrow$

$\{\beta : \mathscr{M}_\beta \models \Psi(v,x)[g(\beta),h(\beta)]\} \in \mathscr{U} \Longrightarrow$

$\{\beta : \mathscr{M}_\beta \models (\exists x)\Psi(v,x)[g(\beta),h(\beta)]\} \in \mathscr{U}$. For the other implication,

$\{\beta : \mathscr{M}_\beta \models (\exists x)\Psi(v,x)[g(\beta),h(\beta)]\} \in \mathscr{U} \Longrightarrow \exists S \in \mathscr{U}$ s.th.

$\beta \in S \Rightarrow S_\beta := \{a : a \in \mathscr{M}_\beta \text{ and } \mathscr{M}_\beta \models \Psi(v,x)[g(\beta),h(a)]\} \neq \emptyset$

By the axiom of choice, we can choose $h \in \prod_{\beta\in I} \mathscr{M}_\beta$ so that $h(\beta) \in S_\beta$

for $\beta \in S$ so letting $\overline{h} := h/\sim$ we have $\mathscr{M} \models \Psi(v,x)[g,\overline{h}]$

hence, $\mathscr{M} \models (\exists x)\Psi(v,x)[g]$ $\square$

Paul Sacawa.
Quillen-Suslin Theorem

Theorem A finitely generated projective module P over a polynomial ring over a field $R = k[x_1, \ldots, x_k]$ is free.

One studies in K-theory the functor $K_0 : \mathrm{Ring} \to \mathrm{Ab}$ given by: for ring R, consider the set of isomorphism classes of finitely generated projective modules over R. This has a natural semigroup structure under $\oplus$, direct sum of modules. Call it $(S_R, +)$. We then take the Grothendieck group

$G(S)$, standard extension of semigroup to abelian group: take equivalence

$\sim$ on $S \times S$ given by $(a, b) \sim (a', b')$ iff $\exists k : a + b' + k = b + a' + k$.

$M \times M \backslash \sim$ has natural group structure, and we define $K_0(R) := G(S_R)$.

For field $R = k$, $S_R = (\mathbb{N}, +)$, because module is determined by dimension

(similar for PID), so $K_0(k) = \mathbb{Z}$. Similarly, Quillen-Suslin theorem states

that $K_0(k[x_1, \ldots, x_n]) = \mathbb{Z}$ for the same reason.

For it, Quillen received Fields' medal.

Tychonoff's Theorem (TT): *an arbitrary product of compact topological spaces is compact in the product topology.*

Def. 1. For X Banach X^* its dual with operator norm: weak* topology of X is the coarsest s.th. $\forall x \in X$, $\{T_x(\phi) := \phi(x)\}_{\phi \in X^*}$ are continuous.

Alaoglu Thm: $\{f : ||f|| \leq 1\} \subset X^*$ is compact in weak* topology.

Def. 2. Let X be topological space, (Y, d) - metric space, Y^X - set of all functions mapping X into Y, and $C_{XY} := \{f \in Y^X : f \ is \ cont.\}$.

Given $f \in Y^X$, $\epsilon > 0$, compact subspace C of X ,

let $B_C(f,\epsilon) := \{g \in Y^X : \ sup\{d(f(x), g(x))| \ x \in C\} < \epsilon \ \}$.

Topology of compact convergence is topology with basis sets $B_C(f,\epsilon)$.

$F \subset C_{XY}$ is *equicontinuous* if, for each $x_0 \in X$, given $\epsilon > 0$,

$\exists$ nbh. U of x_0 s.th. $\forall x \in U$ and $\forall f \in F$, $d(f(x), f(x_0)) < \epsilon$.

Theorem(Ascoli): Given C_{XY} topology of compact convergence,

if $F \subset C_{XY}$ is equicontinuous and $F_a := \{f(a) : f \in F\}$ has compact

closure for each $a \in X$, then F is contained in compact subspace of C_{XY} .

The converse holds if X is locally compact Hausdorff.

Dylan Butson. Induced Representations of H on G .

Theorem *Let G a locally compact topological group with a closed subgroup $H \subset G$ such that G/H has a G-invariant measure μ. Let $\mathcal{H}$ be a Hilbert space $\sigma : H \to U(\mathcal{H})$ a unitary representation of H. Then there exists a natural unitary representation $\pi_{H,\sigma} : G \to U(\mathcal{F})$.*

Viewing G as a principal H bundle over G/H, we can construct the associated vector bundle $E = G \times_\sigma \mathcal{H}$. Let $\mathcal{F}$ to be the space of L^2 sections on this bundle with respect to $|f|^2 = \int_{G/H} |f(\bar{g})|^2_{\mathcal{H}} d\mu$.

Equivalently, first define $\mathcal{F}_0$ as the space of continuous functions

$f : G \to \mathcal{H}$ such that $f(gh) = \sigma(g)^{-1} f(g)$. Then, take the completion of

$\mathcal{F}_0$ with respect to the above norm. Note that the above norm is well

defined on $\mathcal{F}_0$ by the equivariance property above and unitarity of σ.

The representation is then $[\pi(g)f](g') = f(g^{-1}g')$. It is clear that

$\pi(g) : \mathcal{F} \to \mathcal{F}$ for each $g \in G$. Further, this representation is unitary since

$|\pi(g)f|^2 = \int_{G/H} |\pi(g)f(\bar{g}')|^2_{\mathcal{H}} d\mu(g') = \int_{G/H} |f(\overline{g^{-1}g'})|^2_{\mathcal{H}} d\mu(g') =$

$\int_{G/H} |f(g')|^2_{\mathcal{H}} d\mu(g') = |f|^2$ by G-invariance of μ.

Viktoriya Baydina. Pick's Theorem: $A = I + \frac{B}{2} - 1$, where

A = area of a lattice polygon; $I = \#$ of interior lattice points; $B = \#$ of

boundary lattice points (vertices included); elementary triangle (ET) =

vertices are lattice points & has no further boundary or interior points.

Lemma 1: Any lattice polygon can be triangulated by ETs.

Lemma 2: Area of any ET in a $\mathbb{Z}^2$ lattice is $\frac{1}{2}$.

Proof of PT: Partition polygon P into N ETs, by Lemma 1. Idea: sum

internal angles of the ETs in two ways. (1) Angle sum of any triangle is π, so total is $T = N \cdot \pi$. (2) At each interior point i, angles of ETs having i as a vertex sum to 2π. At each non-vertex boundary point b, angles of ETs having b as vertex sum to π. If number of vertices is k, interior angles at the vertices add to $k\pi - 2\pi$ since sum of exterior angles is 2π (walking along perimeter of polygon, one completes a full turn). Thus sum of angles at boundary points is $B \cdot \pi - 2\pi$, and sum of angles at internal points is $I \cdot 2\pi$. Thus $T = I \cdot 2\pi + B \cdot \pi - 2\pi$. (1) + (2) $\implies$ $N = I \cdot 2 + B - 2$. By Lemma 2, $A = N \cdot \frac{1}{2} \implies A = I + \frac{1}{2}B - 1$. ■

Tomas Kojar.
1. Hodge Decomposition Theorem for Kahler Manifolds.

Let M be a compact complex manifold of Kahler type. Then there is a direct sum decomposition: $H^r(M, \mathbb{C}) = \bigoplus_{p,q: p+q=r} H^{p,q}(M)$.

$H^r(M, \mathbb{C})$ is the deRham group of r-forms on M (for related details see Vitali's and Dylan's talks). $H^{p,q}(M)$ is the cohomology of complex differential forms of degree (p,q) on M (called Doulbeaut group).

In local coordinates **Hermitian metric** is $h := \sum h_{ij} dz^i \wedge d\bar{z}^j$, where h_{ij}

are entries of a positive definite Hermitian matrix ($H = \bar{H}^{tr}$). The

Hermitian form is $\omega := \frac{i}{2}(h - \bar{h}) = \frac{i}{2}\sum_{i,j} h_{ij} dz^i \wedge d\bar{z}^j$. A **Kahler**

manifold is a complex manifold with Hermitian metric and the associated

Hermitian form closed ($d\omega = 0$), in which case it is called Kahler metric.

2. Given a Morse-Smale pair on a compact smooth manifold M. Then the homology of the Morse-Smale complex is isomorphic to the singular homology of this manifold: $H^k_{Morse}(f, \mathbb{Z}) \simeq H^k_{sing}(M)$.

About Morse functions see "Morse Generic and Thm" presentation. Next

Stable and Unstable manifolds: Consider for Morse function f on M the flow $\phi : \mathbb{R} \times M \to M$ of the vector field $- \bigtriangledown f(x)$ with respect to a Riemmanian metric g . For critical points p of f (shortly $p \in Cr(f)$):

Stable $W_p^s = \{x \in M : lim_{t \to +\infty} \phi(t,x) = p\}$ and has $dim(W_p^s) = Ind_p f$

Unstable $W_p^u = \{x \in M : lim_{t \to -\infty} \phi(t,x) = p\}$ and has $dim(W_p^u) = dim(M) - Ind_p f$. **Morse-Smale condition** (M-S): For any $p, q \in Cr(f))$, $T_x M = T_x W_p^s + T_x W_p^u$. We say that (f,g) is a **Morse-Smale pair**.

Towards Explaining Morse Homology $H^k_{Morse}(f, \mathbb{Z})$.

A flow line between $p, q \in Cr(f)$ is a path $\gamma : \mathbb{R} \to M$ s.th.

$\gamma(s)' = -\bigtriangledown f(\gamma(s))$, $lim_{s\to-\infty}\gamma(s) = p$ and $lim_{s\to+\infty}\gamma(s) = q$. Let

$M(p, q) := (W^s_q \cap W^u_p)/\mathbb{R}$, i.e. the moduli space of flow lines btw p,q

$\overset{M-S}{\Rightarrow} M(p, q)$ is a submanifold with $dim(M(p, q)) = Ind_p f - Ind_q f - 1$.

Orientation of $M(p, q)$: For each W^u_p we choose an orientation

$\Rightarrow TW^u_p \overset{M-S}{\simeq} T(W^u_p \cap W^s_q) \oplus (TM/TW^s_q) \simeq T_\gamma M(p, q) \oplus T_\gamma \oplus T_q W^u_q$,

where $(TM/TW^s_q) \simeq T_q W^u_q$ follows from translating W^u_q

Morse-Smale complex

to the complement $\Rightarrow$ we pick orientation for $M(p,q)$ accordingly .

Consider the module $C_k(f) := \bigoplus_{p \in Cr_k(f)} \mathbb{Z}[p]$, where

$Cr_k(f) = \{p \in M : p \in Cr(f) \text{ and } Ind_p f = k\}$. And operator:

$\partial^k_{Morse} : C_k \to C_{k-1}$ as $\partial_{Morse}(p) := \sum_{Ind_q(f)=k-1} orient(M(p,q)) \cdot q$.

This gives you an exact sequence

$$0 \to C_n(f) \stackrel{\partial_n}{\to} \ldots C_{k+1}(f) \stackrel{\partial_{k+1}}{\to} C_k(f) \stackrel{\partial_k}{\cdot} .. \stackrel{\partial_2}{\to} C_1(f) \to 0$$

and thus homology $H^k_{Morse}(f, \mathbb{Z}) := \frac{Ker(\partial_k)}{Im(\partial_{k-1})}$.