Whitney's Classification of $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$

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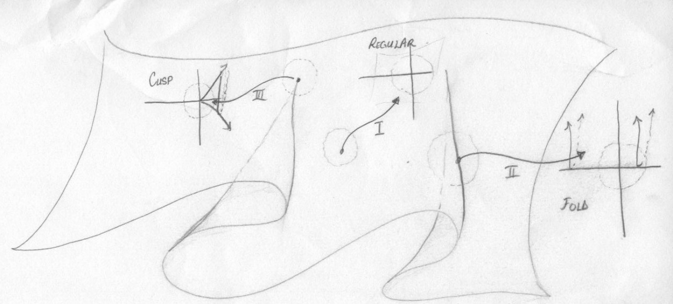
CLASSIFICATION (WHITNEY) OF $F \in T_{open,dense} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^2)$

$\forall a \in \mathbb{R}^2 \ F_a := F : (\mathbb{R}^2, a) \mapsto (\mathbb{R}^2, a) \simeq_{diff} G : (\mathbb{R}^2, 0) \xrightarrow{(id,g)} (\mathbb{R}^2, 0)$

(I): (Regular) $G_1 := (x, z) \iff \frac{\partial g}{\partial z}(a) \neq 0$

(II): (Fold) $G_2 := (x, z^2) \iff \frac{\partial^2 g}{\partial z^2}(a) \neq 0, \frac{\partial g}{\partial z}(a) = 0$, otherwise

(III): (Cusp) $G_3 := (x, z^3 - xz) \iff \frac{\partial^2 g}{\partial x \partial z}(a) \neq 0, \frac{\partial^3 g}{\partial z^3}(a) \neq 0$



Nbds. $\{U(\epsilon, k)\}_{k\in\mathbb{Z}^+,\epsilon\in C^0(\mathbb{R}^n,\mathbb{R}^+)}$ of $0 \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ are

$F \in \{U(\epsilon, k)\} \iff |D^\alpha F_j(x)| < \epsilon(x)$, $j = 1, ..., p\ \forall\ |\alpha| \le k, x \in \mathbb{R}^n$.

Step 1: Maps $G \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $DG(x) \neq 0\ \forall x \in \mathbb{R}^2$ are dense.

Proof Fix nbh. of U of $F \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$.

Sard $\Rightarrow$ measure$(Im\{DF : \mathbb{R}^2 \to \mathbb{R}^4\}) = 0$

$\Rightarrow \forall$ nbd. V of $0 \in Lin_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2) = \mathbb{R}^4$, $\exists \mu \in V$ s.th. $\mu \notin Im(DF)$.

For $\bar{G}(x) := F(x) - \mu \cdot x \Rightarrow D\bar{G}(x) \neq 0\ \forall x$. Set $\phi_{-1} := 0$ and

$B_{(0,n)} := \{x \in \mathbb{R}^2 : ||x|| \leq n\}$, $n \geq 1$, then

$\exists\ \phi_n \in C^\infty(\mathbb{R}^2, \mathbb{R})$ s.th. $\phi_n|_{B_{(0,n)}} = 1$ and $\phi_n|_{\mathbb{R}^2 \setminus B_{(0,n+1)}} = 0$.

For every n choose $\bar{G}_n$ to be near F on $B_{(0,n+1)} \setminus B_{(0,n+1)}$.

Construct $G^n := (\phi_n - \phi_{n-2})\bar{G}_n + (1 - (\phi_n - \phi_{n-2}))G^{n-1}$

Note, G^n and G^{n-1} differ only on $B_{(0,n+1)} \setminus B_{(0,n-2)}$.

For small μ (for $\bar{G}_n$), $G^n \in U$ and $(\frac{\partial G_i^n}{\partial x_j})_{i,j=1,2} \neq 0$ on $B_{(0,n-1)}$.

For G $:= \lim\limits_{n \to \infty} G^n \Rightarrow G \in U \subset C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. $\square$

Corollary. Imp. F. Thm. $\Rightarrow \exists\ T_{open,dense} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^2)$

such that G locally looks like $(x, z) \mapsto (x, \phi(x, z))\ \forall G \in T$.

STEP 2: CLASSIFYING POSSIBLE $\phi(x,z)$

Let $V \subset \mathbb{R}^2$ be open, $K \subset V$ compact, $\phi \in C^\infty(V,\mathbb{R})$,

$\exists\, g \in C^\infty(V,\mathbb{R})$ arbitrarily close to ϕ on K s.th. $\forall\, a \in V$

(I) $\frac{\partial g}{\partial z}(a) \neq 0$, or (II) $\frac{\partial^2 g}{\partial z^2}(a) \neq 0$, or (III) $\frac{\partial^2 g}{\partial x \partial z} \neq 0$ and $\frac{\partial^3 g}{\partial z^3}(a) \neq 0$

Proof $g(x,z) := \phi(x,z) + \lambda_1 z + \lambda_2 z^2 + \lambda_3 xz + \lambda_4 z^3$ with a 'small'

$\Lambda := (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \in \mathbb{R}^4 \Rightarrow \frac{\partial g}{\partial z} = \frac{\partial \phi}{\partial z} + \lambda_1 + 2\lambda_2 z + \lambda_3 x + 3\lambda_4 z^2$

then $\frac{\partial g}{\partial z}|_a = 0 \Leftrightarrow (1, 2z, x, 3z^2)|_a \cdot \Lambda = -\frac{\partial \phi}{\partial z}|_a$,

$\frac{\partial^2 g}{\partial z^2} = \frac{\partial^2 \phi}{\partial z^2} + 2\lambda_2 + 6\lambda_4 z$ then $\frac{\partial^2 g}{\partial z^2}|_a = 0 \Leftrightarrow (0,2,0,6z)|_a \cdot \Lambda = -\frac{\partial^2 \phi}{\partial z^2}|_a$,

$$\frac{\partial^2 g}{\partial x \partial z} = \frac{\partial^2 \phi}{\partial x \partial z} + \lambda_3 \text{ then } \frac{\partial^2 g}{\partial x \partial z}|_a = 0 \Leftrightarrow (0,0,1,0) \cdot \Lambda = -\frac{\partial^2 \phi}{\partial x \partial z}|_a \ ,$$

$$\frac{\partial^3 g}{\partial z^3} = \frac{\partial^3 \phi}{\partial z^3} + 6\lambda_4 \text{ then } \frac{\partial^3 g}{\partial z^3}|_a = 0 \Leftrightarrow (0,0,0,6) \cdot \Lambda = -\frac{\partial^3 \phi}{\partial z^3}|_a \ .$$

$$A := \begin{pmatrix} 1 & 2z & x & 3z^2 \\ 0 & 2 & 0 & 6z \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \bar{A} := \begin{pmatrix} 1 & 2z & x & 3z^2 \\ 0 & 2 & 0 & 6z \\ 0 & 0 & 0 & 6 \end{pmatrix} \text{ are of rank 3}$$

$\Rightarrow$ **Lemma** Measure of $\Lambda \in \mathbb{R}^4$ s.th. $A(a) \cdot \Lambda = B(a) := - \begin{pmatrix} \frac{\partial \phi}{\partial z}(a) \\ \frac{\partial^2 \phi}{\partial z^2}(a) \\ \frac{\partial^2 \phi}{\partial x \partial z}(a) \end{pmatrix}$

(for some a $\in$ V) is 0 .

Proof. Map $\mathcal{A}$: $V \times \mathbb{R}^4 \ni (a, \Lambda) \longmapsto (a, A(a)\Lambda) \in V \times \mathbb{R}^3$ is

submersion due to $rankD\mathcal{A} \equiv 5$! $\quad 3 =$ codim of the following:

$\{(a, B(a)) : a \in V\} \subset V \times \mathbb{R}^3$, $\mathcal{A}^{-1}(\{(a, B(a))|a \in V\} \subset V \times \mathbb{R}^4)$

$\Longrightarrow measure(\pi(\mathcal{A}^{-1}(\{(a, B(a))|a \in V\})) = 0$, $\pi(a, \Lambda) := \Lambda$. $\square$

Similarily, $measure(\pi(\bar{\mathcal{A}}^{-1}(\{(a, B(a))|a \in V\})) = 0$, where

$\bar{\mathcal{A}}$: $V \times \mathbb{R}^4 \ni (a, \Lambda) \longmapsto (a, \bar{A}(a)\Lambda) \in V \times \mathbb{R}^3$. We completed

Step 2 by modifying $\phi(x, z)$ by means of $\Lambda \in \mathbb{R}^4$ to $g(x, z)$.

Corollary Assume $G \in C^\infty(V, R^2)$ and $G(x,z) \mapsto (x, g(x,z))$

where $g(x,z)$ satisfies conditions of conclusion of step 2.

If $\tilde{G} =: (\tilde{G}_1, \tilde{G}_2)$ close to G over compact K in V (suffices up to

3rd order derivatives) then by means of diffeomorphism

$(y_1, y_2) \overset{H}{\mapsto} (\tilde{G}_1, y_2)$, $\tilde{G} \simeq_{diff} (y_1, y_2) \mapsto (y_1, \tilde{g}(y_1, y_2))$ where

$\tilde{g}$ satisifies conditions of conclusion of step 2 similar to those of g

for a in a nbd. of K. This is true because $\tilde{g} = \tilde{G}_2 \circ H^{-1}$ and

therefore is close to g over K (up to 3rd order derivatives).

This shows that the conclusion of step II is a stable condition .

Step 3: $\exists\, T_{open,dense} \subset C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ such that $\forall G \in T$, G locally is of form $(x,z) \mapsto (x,\phi(x,z))$ and ϕ satisfies conditions in Step 2.

Proof Let $\tilde{F} \in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ and W a nbh. of $\tilde{F}$, Step 1 claims

$\exists\, F \in W$ s.th. $\exists$ locally finite $\bigcup_{n\geq 1} U_n = \mathbb{R}^2$ with $\frac{\partial F_i}{\partial x_j}|_{U_n} \neq 0$

$\forall n$ for some fixed $i,j \Rightarrow F|_{U\,n} \simeq_{diff} (x_1,x_2) \mapsto (x_1,\phi_n(x_1,x_2))$.

$\exists \bigcup_n K_n = \mathbb{R}^2$, s.th K_n compact $\subset U_n$. Using corollary

of step 2, every G near F satisfies conclusion of Step 3.

So, pertub F on K_n to obtain

$G_n|_{Un} \simeq_{diff} (x_1, x_2) \mapsto (x_1, g_n(x_1, x_2))$ (i, j fixed g_n as g of step 2) .

Note, G near $F \Leftrightarrow g_n$ close to ϕ_n on $\bar{U}_n$.

Construct $G^n := \Psi_n \cdot G_n + (1 - \Psi_{n-1})G^{n-1}$, $n \geq 1$, $\Psi_0 \equiv 1$,

$\Psi_n|_{K_n} :\equiv 1$ and $\Psi_n|_{U_n^c} :\equiv 0 \Rightarrow$G^n and G^{n-1} differ only on $U_n \setminus U_{n-1}$

Similar to step 1, $G := \lim_{n\to\infty} G^n$ is the desired map . DONE.

Step 4: $\forall F \in T$, F is diffeo. locally to Regular, Fold or Cusp.

Proof Say $\mathcal{E}^2_{(2)} \ni F : (x,z) \mapsto (x, f(x,z))$ is a C^∞ germ at 0 s.th.

(I) $\frac{\partial f}{\partial z}(0) \neq 0$ or (II) $\frac{\partial f}{\partial z}(0) = 0$ and $\frac{\partial^2 f}{\partial z^2}(0) \neq 0$ or

(III) $\frac{\partial f}{\partial z}(0) = \frac{\partial^2 f}{\partial z^2}(0) = 0$, $\frac{\partial^2 f}{\partial x \partial z}(0) \neq 0$ and $\frac{\partial^3 f}{\partial z^3}(0) \neq 0$.

(I) (Regular): (Inv. M. Thm) $\Rightarrow F \simeq_{diff} (x,z) \mapsto (x,z)$

(II) (Fold): $\hat{f}(0,z) = f(0,0) + \frac{\partial f}{\partial z}(0)z + \frac{\partial^2 f}{\partial z^2}(0)z^2 \bmod \hat{m}^3_{(1)} \subseteq \hat{\mathcal{E}}_{(1)}$

$\Rightarrow f(0,z) = z^2 q(z)$ with $q(0) \neq 0$ and using

$f(x,z) = f(0,z) + x \cdot r(x,z) \Rightarrow \mathcal{E}_{(2)} \supset (x,f) = (x,z^2)$.

With $\mathcal{E}_{(2)} \ni g \mapsto \hat{g} \in \hat{\mathcal{E}}_{(2)}$ being Taylor homomorphism let

$\hat{g}(x,z) =: g(0) + g_1 x + g_2 z + g_3 xz + g_4 z^2 \text{ mod } (\hat{m}^3_{(2)}, x^2) \subseteq \hat{\mathcal{E}}_{(2)}$

$\Rightarrow \hat{g}(x,z) = g(0) + g_2 z \text{ mod } (x,z^2) \Rightarrow \hat{\mathcal{E}}_{(2)}/\hat{F}^*\hat{m}_{(2)} = <1,z>_{\mathbb{R}}$

$\Rightarrow <1,z>_{F^*\mathcal{E}_{(2)}} = \mathcal{E}_{(2)}$, i.e. 1 and z generate $\mathcal{E}_{(2)}$ as an $F^*\mathcal{E}_{(2)}$-module

$\Rightarrow z^2 = \Phi(x, f(x,z)) \cdot 1 + 2\Psi(x, f(x,z)) \cdot z$, $\Phi, \Psi \in \mathcal{E}_{(2)}$,

$\Rightarrow \Phi(0) = \Psi(0) = 0$, $\frac{\partial \Phi}{\partial y}(0) \neq 0$ and

$\frac{\partial\Psi(x,f(x,z))}{\partial z}(0) = \frac{\partial\Psi}{\partial y}(0) \cdot \frac{\partial f}{\partial z}(0) = 0$. Therefore h and k with

$$h(x,z) := (x, z - \Psi(x, f(x,z))),\ k(x,y) := (x, \Phi(x,y) + \Psi^2(x,y))$$

are diffeomorphisms of $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^2, 0)$. Note that

$$z^2 + (F^*\Psi)^2 - 2F^*\Psi \cdot z = F^*\Phi + 2F^*\Psi \cdot z + (F^*\Psi)^2 - 2F^*\Psi \cdot z$$

$= F^*\Phi + (F^*\Psi)^2 \Longrightarrow$ F is a Fold: by the commutatively of

$$\begin{array}{ccc}(x,z) & \overset{h}{\longmapsto} & (x,u) := (x, z - \Psi(x, f(x,z))) \\ \big\downarrow F & & \big\downarrow \\ (x, f(x,z)) & \overset{k}{\longmapsto} & (x,u^2) := (x, \Phi(x, f(x,z)) + \Psi^2(x, f(x,z)))\end{array}$$

(III) (Cusp): As in II (see slide 19) $\Rightarrow \exists$ germs $\tilde{\Phi}, \tilde{\Psi}, \tilde{\Theta}$ s.th.

$$z^3 = \tilde{\Phi}(x, f(x,z)) \cdot 1 + \tilde{\Psi}(x, f(x,z)) \cdot z + 3\tilde{\Theta}(x, f(x,z)) \cdot z^2 \ ,$$

$$\Rightarrow \tilde{\Phi}(0) = \tilde{\Psi}(0) = \tilde{\Theta}(0) \ , \quad T_0^3(F^*\tilde{\Theta}|_{x=0}) = 0 \Rightarrow$$

$$\begin{array}{ccc}(R^2, 0) \ni (x,z) & \overset{(id,\phi)}{\longmapsto} & (x, \bar{z}) := (x, z - \tilde{\Theta}(x, f(x,z))) \\ & \searrow f \quad \swarrow f_T & \\ & f(x,z) = f_T(x, \bar{z}) & \end{array}$$

Modulo $\hat{m}^3_{(2)} \subset \hat{\mathcal{E}}_{(2)}$ it follows:

(i) $\hat{f}(x,z) =: f_1 x + f_2 x^2 + f_3 xz$ (ii) $\hat{\phi}(x,z) =: z + c_1 x + c_2 x^2 + c_3 xz$

$\Rightarrow$ (iii) $\hat{\phi}^{-1}(x,\bar{z}) = \bar{z} - c_1 x - (c_2 - c_1 c_3)x^2 - c_3 x\bar{z}$ and

(iv) $\hat{f}_T(x,\bar{z}) = \hat{f}(x,\phi(x,\bar{z})) = f_1 \cdot x + (f_2 - f_3 c_1) \cdot x^2 + f_3 x\bar{z}$

$\Rightarrow \frac{\partial^k}{\partial \bar{z}^k} f_T(0,\bar{z}) = 0 \forall k \leq 2$

and also $\frac{\partial f_T}{\partial x}(0,\bar{z})$ vanishes exactly to 1^{st} order.

$\Rightarrow$ We may assume $\tilde{\Theta} \equiv 0$ (see slide 20) , i.e. $z^3 = F^*\Phi \cdot 1 + F^*\Psi \cdot z$ and $\Phi(0,0) = \Psi(0,0) = 0$.

Lemma.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} \Psi(x_1, f(x_1, x_2)) \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \overset{k}{\mapsto} \begin{pmatrix} \Psi(y_1, y_2) \\ \Phi(y_1, y_2) \end{pmatrix}$$

are coordinate changes in the source and target.

Proof $\hat{f}(x_1, x_2) =: f_1 x_1 + f_2 x_1^2 + f_3 x_1 x_2 + f_4 x_2^3 \quad \bmod (\hat{m}^4_{(2)}, x_1^3)$

with $f_3, f_4 \neq 0$ by assumption in (III) . Modulo $\hat{m}^2_{(2)}$ let

$\hat{\Phi}(x_1, y) =: b_1 x_1 + b_2 y$, $\hat{\Psi}(x_1, y) =: c_1 x_1 + c_2 y$.

Recall $x_2^3 = \hat{\Phi}(x_1, \hat{f}(x_1, x_2)) + \hat{\Psi}(x_1, \hat{f}(x_1, x_2)) \cdot x_2 \Rightarrow x_2^3 = b_2 f_4 x_2^3 +$

$(b_1 + b_2 f_1)x_1 + b_2 f_2 x_1^2 + (b_2 f_3 + c_1 + c_2 f_1)x_1 x_2 + c_2 f_2 x_1^2 x_2 + c_2 f_3 x_1 x_2^2$

$\Rightarrow b_2 \neq 0$ since $b_2 f_4$ is coeff. at x_2^3 . Since $(b_2 f_3 + c_1 + c_2 f_1) = 0$

$\Rightarrow c_1 + c_2 f_1 \neq 0$ (so h is a diffeo.) . Now $(b_1 + b_2 f_1) = 0$ and

$\det|(b_i, c_j)| = b_1 c_2 - b_2 c_1 = -b_2(f_1 c_2 + c_1) \neq 0 \Rightarrow$ k is a diffeo. $\square$

Recall $x_2^3 - F^*\Psi \cdot x_2 = F^*\Phi$, where $x_2 := \bar{y} \Rightarrow$ commutativity of

$$\begin{array}{ccc} (x,z) & \longmapsto & (\Psi(x,f(x,z)),z) = (\bar{x},\bar{y}) \\ \Big\downarrow F & & \Big\downarrow \\ (x,f(x,z)) & \longmapsto & (\Psi(x,f(x,z)),\Phi(x,f(x,z))) = (\bar{x},\bar{y}^3 - \bar{x}\bar{y}) \end{array}$$

$\Rightarrow F \simeq_{diff} (x,z) \mapsto (x, z^3 - xz)$, which completes proof of the result

modulo several calculations presented in the pages following.

Appendix (I) Malg. Prep. Thm case of Cusp

$$\hat{f}(0,z) = f(0) + \frac{\partial f}{\partial z}(0)z + \frac{\partial^2 f}{\partial z^2}(0)z^2 + \frac{\partial^3 f}{\partial z^3}(0)z^3 \bmod \hat{m}^4_{(1)} \subseteq \hat{\mathcal{E}}_{(1)}$$

and in the 'Cusp' case $\Rightarrow f(0,z) = g(z) \cdot z^3$, $g(0) \neq 0$.

Therefore $f(x,z) = f(0,z) + x \cdot r(x,z) \Rightarrow \mathcal{E}_{(2)} \supset (x,f) = (x,z^3)$.

By the Taylor homomorphism set $\forall g \in \mathcal{E}_{(2)}$

$\Rightarrow \hat{g}(x,z) = g(0) + g_2 z + g_3 z^2 \bmod (x,z^3) \subset \mathcal{E}_{(2)}$, g_2 and $g_3 \in \mathbb{R}$

$\Rightarrow \hat{\mathcal{E}}_{(2)}/\hat{F}^*\hat{m}_{(2)} =< 1, z, z^2 >_{\mathbb{R}} \Rightarrow < 1, z, z^2 >_{F^*\mathcal{E}_{(2)}} = \mathcal{E}_{(2)}$

i.e. 1, z, z^2 generate $\mathcal{E}_{(2)}$ as an $F^*\mathcal{E}_{(2)}$-module . $\square$

Appendix (II) Justifying assumption $\tilde{\Theta} \equiv 0$

$$\bar{z}^3 = (z - F^*\tilde{\Theta})^3 = z^3 - 3z^2 \cdot F^*\tilde{\Theta} + 3z \cdot F^*\tilde{\Theta}^2 - F^*\tilde{\Theta}^3$$

$$= F^*\tilde{\Phi} + F^*\tilde{\Psi} \cdot z + 3F^*\tilde{\Theta}z^3 - 3z^2 F^*\tilde{\Theta} + 3zF^*\tilde{\Theta}^2 - F^*\tilde{\Theta}^3$$

$$= (F^*\tilde{\Psi} + 3F^*\tilde{\Theta}^2)(z - F^*\tilde{\Theta}) + (F^*\tilde{\Phi} + 2F^*\tilde{\Theta}^3 + F^*\tilde{\Psi} \cdot F^*\tilde{\Theta})$$

$=: F^*\tilde{\Psi}_1 \cdot \bar{z} + F^*\tilde{\Phi}_1 \cdot 1$, which shows that we may assume to

begin with that $\tilde{\Theta} \equiv 0$ without loss of generality, as required . $\square$