Chow's theorem. Proof and Applications.

Mitsuru Wilson
University of Toronto

March 26, 2010

$\mathbb{C}$ **analytic set** A **in** $\Omega \subset \mathbb{C}^n$ **is locally the set** $V\left(\bar{f}\right)$

of zeros of holomorphic functions $\bar{f} = (f_1, \ldots, f_k)$

Reg $A := \{a \in A : (A, a) \text{ is smooth}\}$,   Sing$A := A -$Reg A.

Thm. A analytic $\Longrightarrow$ Sing$A$ analytic and $\dim(\mathrm{Sing}A) < \dim A$.

Cor. A analytic $\Longrightarrow$ A *analytic, i.e., $A = \bigcup_{0 \leqq l \leqq d} A^l$ where

each A^l smooth analytic in $\Omega - \bigcup_{0 \leqq j < l} A^j$ of $\dim A^l = l$

and $\bigcup_{0 \leqq j \leqq l} A^j$ is a closed set for every $0 \leqq l \leqq d$.

Main Thm. A **is *analytic** $\Longrightarrow$ $\overline{A^d}$ **is analytic**

Cor 1. $\exists$ loc finite $A = \bigcup_{A_k \in \text{irred}} A_k$, ($A$ irred $\Longleftrightarrow$ $A \neq A_1 \cup A_2$)

Set $A - \text{Sing}A =: \bigcup_j X_j$ with X_j smooth conn. analytic of dim A

$\Longrightarrow$ $\bar{X}_j - X_j \subset \text{Sing}A$ $\Longrightarrow$ by thm, $\bar{X}_j$ is irreducible $\square$

Cor 2 (Chow's Thm). Every analytic $A \subset \mathbb{P}^n$ is algebraic.

Proof. Let $\pi : \mathbb{C}^{n+1} - \{0\} \ni (z_0, \ldots, z_n) \mapsto [z_0 : \ldots : z_n] \in \mathbb{P}^n$

$\Rightarrow$Cone $Z := \pi^{-1}(A) \cup \{0\}$ is *analytic $\Longrightarrow$ Z analytic $= V(\bar{f})$.

$$f=\sum_{|\alpha|\geqq 1} c_{f,\alpha} z^{\alpha} \,|_{Z} \equiv 0 \implies f(\lambda z)=\sum_{r=1}^{\infty} \lambda^{r} \sum_{|\alpha|=r} c_{f,\alpha} z^{\alpha} \equiv 0 \;\forall\, |\lambda| \leqq 1$$

$$\implies A=V\left(\ldots, \textstyle\sum_{|\alpha|=\sigma} c_{f,\alpha} z^{\alpha}, \ldots\right) \implies \Box \text{(Hilbert Basis Thm)}$$

Key Lemma. $\pi:\mathbb{C}^{n+q} \ni (\xi,\eta) \mapsto \xi \in \mathbb{C}^{n}$ and $\pi\,|_{A}$ is proper

$\implies \pi(A)$ is analytic and $\pi\,|_{A}$ is finite to one.

Proof. Reduction to $q=1$ case is simple: Factor π as

$$\begin{array}{ccccc} \mathbb{C}^{n+q} & \xrightarrow{\pi_1} & \mathbb{C}^{n+q-1} & \xrightarrow{\pi_2} & \mathbb{C}^{n} \\ (z_1,\ldots,z_{n+q}) & \mapsto & (z_1,\ldots,z_{n+q-1}) & \mapsto & (z_1,\ldots,z_n)=z \end{array}$$

$$\begin{array}{ccccc} & & A \subset \Omega & & \\ & \swarrow^{\pi_1|_A} & & \searrow^{\pi|_A} & \\ U := \pi_1(\Omega) & & \xrightarrow{\pi_2} & & V := \pi_2(\pi_1(\Omega)) \end{array} .$$

$q = 1$ **case:** $w \in V \implies A \cap \pi^{-1}(w)$ compact analytic in

$\Omega \cap \pi^{-1}(w) \subset \mathbb{C} \Rightarrow A \cap \pi^{-1}(w) = \{a_1, \ldots, a_m\}$ discrete. Say U_j

disjoint nbhs of a_j and V small enough s.th. $A \cap \pi^{-1}(V) \subset \bigcup_j U_j$

Restrict to nbh U_j of a_j, say $a_j = 0$ and $t = (t_1, \ldots, t_{k-1}) \in \mathbb{C}^{k-1}$

Lemma. $\pi(A \cap \pi^{-1}(V) \cap U_j)$ is anal in V, $A \cap \pi^{-1}(w) = \{0\}$

Proof. Note $\exists$ coord. s.th. $d := \mathrm{ord}_0 f_k(0, z_{n+1}) < \infty$

$\Longrightarrow$ by Weirstrauss Prep. and Division Thms ,

$f_k = z_{n+1}^d + \sum a_l z_{n+1}^{d-l}$, $f_j = \sum_{l<d} b_{l,j} z_{n+1}^{d-j}$, with $a_l, b_{l,j} \in \mathbb{C}\{z\}$.

$\mathbb{C}\{z\}[t] \ni \mathrm{Res}(f_k, f_k + t_1 f_1 + \cdots + t_{k-1} f_{k-1}) = \sum_{|\alpha|=d} t^\alpha R_\alpha$,

Say $V \ni 0$ small enough s.th. all coeff. and R_α holom in V and

for $z \in V$, all roots of $f_k(z, z_{n+1})$ are in U. Put $U' := U \cap \pi^{-1}(V)$

Claim. $\pi(A \cap U \cap \pi^{-1}(V))$ are common roots of R_α in V .

If $(z, z_{n+1}) \in A \cap U' \Rightarrow \forall t \in \mathbb{C}^{k-1} \exists$ a common for $\{f_j(z, z_{n+1})\}$

root $z_{n+1} = \zeta_{n+1} \implies \sum t^\alpha R_\alpha(z_1, \ldots, z_n) = 0 \, \forall \, t \in \mathbb{C}^{k-1}$

$\implies R_\alpha(z) = 0 \, \forall \, \alpha$. Conversely, if $z \in V$ and $R_\alpha(z) = 0 \; \forall \; \alpha$

$\implies \forall \; t \in \mathbb{C}^{k-1} \exists$ common root $\zeta_j \in \mathbb{C}$,

$0 \leqq j \leqq d$, of $f_k(z, \zeta_j) = \sum_{l=1}^{k-1} t_l f_l(z, \zeta_j) = 0$.

Say $W_j := \{t \in \mathbb{C}^{k-1} : \sum_{l=1}^{k-1} t_l f_l(z_1, \ldots, z_n, \zeta_j) = 0\}$

$\implies W_1 \cup \cdots \cup W_d = \mathbb{C}^{k-1} \implies \exists j' \, \text{s.th.} \, W_{j'} = \mathbb{C}^{k-1}$

$\implies f_l(z, \zeta_{j'}) = 0 \ \forall \ l = 1, \ldots, k \implies \square$

The setting: A_1 analytic in Ω, A_0 analytic in $\Omega - A_1$

Main claim. $d := \dim A_0 > \dim A_1$, $\overline{A_0} - A_0 \subset A_1 \Rightarrow A_0$ anal.

Say, $0 \in A_1$, $p \in \Omega - A_0 \cup A_1$, $l \cong \mathbb{C}$ a line from 0 to p

$\implies A_1 \cap l$ analytic in $\Omega \cap l \subset \mathbb{C}$

$\implies \exists$ bounded nbh U of 0 s.th. $A_1 \cap l \cap U = \{0\}$

$\implies$ analytic in $\Omega \cap l - A_1 \cap l$ set $A_0 \cap l$ is discrete off $A_1 \cap l$

$\implies A_0 \cup A_1 \cap l \cap U$ is compact.

Step 2. Let $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ be projection s.th. $l = \pi^{-1}(0)$.

Mumford's lemma (Mustazee) compactness of $\pi|^{-1}_{A_0 \cup A_1 \cap U}(0)$

$\implies \exists$ nbhs $U' \subset U$, $V \subset \mathbb{C}^{n-1}$ of 0 s.th. $\pi \mid_{A_0 \cup A_1 \cap U'}$ is proper.

Key L$\Rightarrow \pi(A_1 \cap U') = B_1$ anal in V , $\pi(A_0 \cap U' - \pi^{-1} B_1) = B_0$

anal in $V - B_1$, and $\pi \mid_{A_1 \cap U'}$ and $\pi \mid_{A_0 \cap U' - \pi^{-1} B_1}$ are finite to one

and each fibre $\pi \mid^{-1}_{A_0 \cup A_1 \cap U} (y)$ for $y \in B_0 \cup B_1$ is countable.

Keep projecting until image of $A_0 \cup A_1$ is a nbh of $\mathbf{0} \in \mathbb{C}^\mu$

$\Rightarrow \mu = d$ since fibers $\pi\mid_{A_0 \cup A_1 \cap U}^{-1}(y)$, $y \in (\mathbb{C}^\mu, 0)$, are countable

$$\begin{array}{ccccc} A_0 - \pi^{-1}(B_1) & \subset & A_0 \cup A_1 & \supset & A_1 \\ \downarrow & & \downarrow \pi & & \downarrow \\ V - B_1 & \subset & V & \supset & \pi(A_1) = B_1 \\ \text{open dense in V} & & \text{nbh, of 0 in } \mathbb{C}^d & & \text{nontrivial anal in V} \end{array}$$

Say, V is a ball $\implies V - B_1$ is connected,

Very important: $\pi\mid_{A_0 - \pi^{-1}(B_1)}$ is proper and finite to one !!!

Let $J(z)$ be the Jacobian of $\pi \mid_{A_0-\pi^{-1}(B_1)}$, then

$J(z) \neq 0 \iff \pi \mid_{A_0-\pi^{-1}(B_1)}$ is an isomorphism near z .

$B := \{z \in A_0 - \pi^{-1}(B_1) : J(z) = 0\}$

$\implies \tilde{B} = \pi(B)$ analytic in $V - B_1$ (Key Lemma)

$z \notin \tilde{B} \iff \pi \mid_{A_0-\pi^{-1}(B_1)}$ is unfamified over z .

Sard Thm$\Rightarrow$reg values of $\pi \mid_{A_0-\pi^{-1}(B_1)}$ are open dense in $V - B_1$

$\implies \tilde{B}$ is a proper analytic subset of $V - B_1$ (Key Lemma).

V and A smooth$\Rightarrow \overline{V - \tilde{B} - B_1} = V$ and $\overline{A_0 - \pi^{-1}(\tilde{B} \cup B_1)} = A_0$

$$\rho := \pi \mid_{A_0 - \pi^{-1}(\tilde{B} \cup B_1)} \text{ then } \begin{array}{ccc} A_0 - \pi^{-1}(B_1 \cup B) & \xrightarrow{\rho} & V - B_1 - B \\ \cap & & \cap \\ \mathbb{C}^n & \xrightarrow{\pi} & \mathbb{C}^d \end{array}$$

Here, ρ finite unramified covering. Next: construction of

holomorphic functions on $\pi^{-1}(V)$ vanishing exactly on $\overline{A_0} \subset \Omega$.

Say $\delta :=$ # of sheets in ρ and λ is linear on $\mathbb{C}^{n-d} := \pi^{-1}(0)$.

$$x^{\delta} + \sum_{1 \leqq l \leqq \delta} \sigma_l(\eta)\, x^{\delta - l} := \prod_{1 \leqq j \leqq \delta} (x - \eta_j)\ \forall\, l \leqq \delta \text{ and } w \in V \setminus B_1 \cup \tilde{B},$$

$$a_l(w) := \sigma_l(\lambda(b_1), \ldots, \lambda(b_\delta)) \text{ with } \rho^{-1}(w) = \{b_1, \ldots, b_\delta\}.$$

Note: $a_l(w)$ are bdd locally on V and holomorphic on $V - B_1 - \tilde{B}$

(as sums of locally holomorphic globally defined functions)

Riemann Ext Thm $\implies$ a_l holomorphically continue to V.

Let $(u, w) := z$ with $w := \pi(z)$ and $u \in \mathbb{C}^{n-d}$ and

$$F_\lambda(z) := \lambda(u)^{\delta} + \sum_{1 \leqq l \leqq \delta} a_l(w) \lambda(u)^{\delta - l}$$

$\implies F_\lambda \equiv 0$ on $A_0 - \pi^{-1}(B_1 \cup B) \Rightarrow F_\lambda \equiv 0$ on $\overline{A_0}$.

$\implies \overline{A_0} \subset \{z \in \pi^{-1}(V) : F_\lambda \equiv 0 \,\forall\, \lambda\}$. Conclusion:

Lemma. Finitely many F_λ suffices to identify $\overline{A_0} \subset \pi^{-1}(V)$.

Proof. Say $z = (u, w) \in \pi^{-1}(V) - \overline{A_0}$ and $w = \lim_{p\to\infty} w_p$

with $w_p \in V - \tilde{B} - B_1 \Rightarrow \rho^{-1}(w_p) = \{b_p^{(1)}, \ldots, b_p^{(\delta)}\}$

By properness and choosing a subsequence$\Rightarrow b_p^{(j)} \to b^{(j)} \,\forall\, j$

Since $b^{(j)} \in \overline{A_0} \implies u \neq b^{(j)} \,\forall\, j = 1, \ldots, \delta$.

Choose a collection $\{*\}$ of $(n-d-1)\delta+1$ of λ's in $(\mathbb{C}^{n-d})^{dual}$

with any $n-d$ of them being linearly independent $\Longrightarrow$

$\exists \lambda \in \{*\}$ s.th. $\lambda(u) \neq \lambda(b^{(j)})$ (on slide 16) , but

$$\{\lambda(b^{(1)}), \ldots, \lambda(b^{(\delta)})\} = \{\xi \in \mathbb{C} : \xi^d + \sum_{1 \leqq l \leqq \delta} a_l(w)\xi^{\delta-l} = 0\}$$

Since $\forall\ p\ \{\lambda(b_p^{(j)})\}_{1 \leqq j \leqq \delta} = \{\xi \in \mathbb{C} : \xi^d + \sum_{1 \leqq l \leqq \delta} a_l(w_p)\xi^{\delta-l} = 0\}$

$$\Longrightarrow\ \overline{A_0} = \{z \in \pi^{-1}(V) : F_\lambda(z) = 0\ \forall\ \lambda \in \{*\}\}\Box$$

Appendix I-exercise

Say collection of $\{*\}$ is picked in as on slide 15. Then, for any

size δ collection $\{b^{(j)}\}_{1 \leqq j \leqq \delta} \subset \mathbb{C}^{n-d}$ and one more point $b \in \mathbb{C}^{n-d}$

$b \in \{b^{(j)}\} \iff \lambda(b) \in \{\lambda(b^{(j)})\}_{1 \leqq j \leqq \delta} \, \forall \, \lambda \in \{*\}$

Appendix II

Riemann Ext Thm. $B \subset \Omega \subset \mathbb{C}^n$ analytic, h holom on $\Omega - B$

and bdd at each $b \in B \implies h$ holomorphic on Ω

Proof. Say φ nonzero holomorphic near $b =: 0$ and $\varphi \mid_B \equiv 0$.

W.P.Thm$\Rightarrow$may assume $\varphi \in \mathbb{C}\{\tilde{z}\}[z_n]$ $\tilde{z} = (z_1, \ldots, z_{n-1})$ and

$\varphi(0, z_n) = z_n^d$ has no multiple factors$\Rightarrow$ $\Delta \not\equiv 0$, where

$\Delta \in \mathbb{C}\{\tilde{z}\}$ is discr of φ in z_n . Say $B' := \{\varphi = 0\} \Rightarrow h \cdot \varphi^2 \in C^0$

and $h \cdot \varphi^2$ is $\mathbb{C}$ 1 time diff $\Rightarrow h \cdot \varphi^2$ is holomorphic

W.Div.Thm$\Rightarrow$ $f := h \cdot \varphi^2 = Q \cdot \varphi + \sum_{j=1}^{d} c_j(\tilde{z}) z_n^{d-j}$ with

$f - Q \cdot \varphi \mid_{B'} = 0 \Rightarrow c_j \equiv 0$ using $\Delta \not\equiv 0$. Can repeat$\Rightarrow$ h is holom .

W.P. Thm. $\mathbb{C}\{t,z\} \ni h(t,z)$, $h(t,0) \approx t^d \Rightarrow h(t,z) \approx P(t,\lambda(z))$

$P(z,\lambda) := t^d + \sum_{j=1}^{d} \lambda_j t^{d-j}$ and $\approx$ means up to a unit factor .

Special Div. Thm and General Div Thm: replace C^∞ by

holomorphic in Omar's talk . Proofs of Spec. Div. Thm $\Longrightarrow$

Weir. Prep. Thm $\Longrightarrow$ General Div. Thm are similar .

But the proof of Spec Div Thm ($\forall\ d \in \mathbb{N}$) is easier :

$\nabla^0 h := h$, $\nabla^1 h(t,s) := \frac{h(t)-h(s)}{t-s}$ etc . Then,

$$(*)\ h(t) - \sum_{j=0}^{d-1} \nabla^j h(s_1, \ldots, s_{j+1}) \cdot (t - s_1) \cdot \ldots \cdot (t - s_{j+1})$$

$$\equiv \nabla^d h(s_1, \ldots, s_d, t) \cdot \prod_{1 \leqq j \leqq d} (t - s_j) \text{ is immediate (induction on } d) \ .$$

$$\sum_{j=0}^{d-1} r_j(s_1, \ldots, s_d) \cdot t^j := \sum_{j=0}^{d-1} \nabla^j h(s_1, \ldots, s_{j+1}) \cdot \prod_{1 \leqq j \leqq d} (t - s_j) \Rightarrow$$

all $r_j \in \mathbb{K}\{s\}$, $\nabla^d h \in \mathbb{K}\{s, t\}$ and are symmetric $\Rightarrow$ (simple)

$\exists R_j(\lambda) \in \mathbb{K}[[\lambda]]$ and $Q_d \in \mathbb{K}[[\lambda, t]]$ s.th. $P(t, \lambda(s)) := \prod_{1 \leqq j \leqq d} (t - s_j)$

$$\Rightarrow \nabla^d h(s, t) \equiv Q_d(\lambda(s), t) \text{ and } r_j = R_j(\lambda(s)) \Rightarrow$$

$$(**)\ h(t) - \sum_{j=0}^{d-1} R_j(\lambda(s)) \cdot t^j - Q_d(\lambda(s), t) \cdot P(t, \lambda(s)) \equiv 0$$

Lemma. For any $\varphi : (\mathbb{K}^n, 0) \ni s \mapsto \lambda(s) \in (\mathbb{K}^n, 0)$ s.th.

$$\max_s(\text{rank}\frac{\partial\varphi}{\partial s}(s)) = n \implies$$

a) $F \in \mathbb{K}[[\lambda]]$, $F \circ \varphi \in \mathbb{K}\{s\} \Rightarrow F \in \mathbb{K}\{\lambda\}$;

b) $F \circ \varphi \equiv 0 \Rightarrow F \equiv 0$ (easy to show) .

Corollary. Obviously, lemma and (**)$\Rightarrow$ Spec Div Thm , i.e.

$$h(t) = Q_d(\lambda \cdot P(t, \lambda) + \sum_{j=1}^{d-1} R_j(\lambda) \cdot t^j \quad \square$$