Algebraic Curves and Riemann Surfaces

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Plane $\mathbb{C}$-Algebraic Curves

$[\xi_0 : \xi_1 : \xi_2]$ homog coord. on $\mathbb{CP}^2$

Projective Curves:

$C := \{ F(\xi_0, \xi_1, \xi_2) = 0\} \subset \mathbb{CP}^2$, F homog. reduced $\deg C = \deg F$

Affine: $C_0 := \{f(x,y) = 0\} \subset \mathbb{C}^2$, reduced $f \in \mathbb{C}[x,y]$ $\deg C_0 = \deg f$

$$p \in sing(C) \Leftrightarrow \left.\frac{\partial F}{\partial x}\right|_p = \left.\frac{\partial F}{\partial y}\right|_p = \left.\frac{\partial F}{\partial z}\right|_p = 0 \qquad \left(\left.\frac{\partial f}{\partial x}\right|_p = \left.\frac{\partial f}{\partial y}\right|_p = 0 \quad \text{affine case} \right).$$

Canonical embedding $\mathbb{C}^2 \hookleftarrow C_0 \ni (x,y) \longmapsto [x : y : 1] \in C \hookrightarrow \mathbb{CP}^2$

Irreducible curves $\longleftrightarrow$ irreducible polynomials

Riemann surfaces C are Connected $\mathbb{C}$ 1-Manifolds

Riemann surfaces are oriented 2-manifolds. Topological classification

of compact Riemann surfaces by (topological) genus $g = 0,1, \ldots$

Theorem(Plücker): For smooth Curves $g = \frac{(d-1)(d-2)}{2}$

with notation degree d and genus g for the respective curve.

Let $P(y,a) := y^p + \sum_{i=1}^{p} a_i y^{p-i}$ and $Q(y,b) := y^q + \sum_{j=1}^{q} b_j y^{q-j}$

The Resultant of $P(y,a)$ and $Q(y,b)$

$D(y,c) := y^d + \sum_{k=1}^{d} c_k y^{d-k} := P(y,a) \cdot Q(y,b) \quad (d = p + q)$

define map φ: $\mathbb{C}^p \times \mathbb{C}^q \ni (a,b) \longmapsto c \in \mathbb{C}^d$ then $\frac{\partial}{\partial a_i}$ and $\frac{\partial}{\partial b_j}$ $\Longrightarrow$

$\sum_{k=1}^{d} \frac{\partial c_k}{\partial a_i} y^{d-k} = y^{p-i} Q(y,b)$ and $\sum_{k=1}^{d} \frac{\partial c_k}{\partial b_j} y^{d-k} = y^{q-j} P(y,a)$

i.e. $Res_{P,Q} \coloneqq \frac{\partial c}{\partial(a,b)} : \mathcal{P}^{p-1} \times \mathcal{P}^{q-1} \ni (F,G) \longmapsto F \cdot Q + G \cdot P$

$res_{P,Q}(a,b) \coloneqq \det Res_{P,Q}(a,b)$, Discriminant: $\mathcal{D}_P(a) \coloneqq res_{P,P'_y}(a)$

Note: $\exists (F,G) \neq 0, Res_{P,Q}(F,G) = 0 \quad iff \quad res_{P,Q}(a,b) = 0.$

Cor. 1: P and Q have a common root $\quad iff \quad res_{P,Q}(a,b) = 0.$

Cor. 2: $\nexists$ common root $P(y,a_0), Q(y,b_0) \Rightarrow \exists \varphi^{-1}\ at\ c_0 \coloneqq \varphi(a_0,b_0)$

Claim: $\mathbb{C}[x,y] \ni P(y,a(x))$ irred $\Rightarrow \mathcal{D}_P(a(x)) \not\equiv 0$ (see Slide 15).

Cor. 3: $sing\big(V(P)\big) \hookrightarrow V(P) \cap \pi^{-1}(\{\mathcal{D}_P(x) = 0\})$ and

is finite if P reduced, where π is the projection onto the x-axis.

Irred. Curves “are” Compact Riemann Surfaces

Cor. 3 plus Chow’s Theorem (Mitsuru’s talk) $\Longrightarrow C^* := C\backslash\mathrm{sing}(C)$ is

connected whenever C irred. ie. is a Riemann surface.

When $sing(C) \neq \emptyset$ $\exists\tilde{C}$ and $\sigma\colon \tilde{C} \to C \hookrightarrow \mathbb{CP}^2$ proper holomorphic s.th.

1) $\sigma(\tilde{C}) = C$ *2)* $\sigma^{-1}(sing(C))$ finite

3) $\sigma\colon \tilde{C}\backslash\sigma^{-1}(sing(C)) \to C^*$ biholomorphic.

By desingularization (Will/Ilia’s talks)

Most important: $\tilde{C}$ is a compact Riemann surface

Holomorphic $f: C \to C'$ of Riemann Surfaces

Prop: $\forall p \in C \; \exists$ local coordinates s.th. not constant $f: w = z^m$

$mult_p(f) \coloneqq m$ unique ; $if \; m > 1, p$ ramification $f(p)$ branch points.

Proof: say $f: w = h(z) \in \mathbb{C}\{z\}$ in local coord centred at p and $f(p)$

$\Rightarrow h(z) = (a \cdot z)^m \cdot g(z)$ with $g(0) = 1, \; g \in \mathbb{C}\{z\} \Longrightarrow$

$z \longmapsto a \cdot z \cdot g(z)^{1/m}$ is a coord. change $\Longrightarrow$ the result. ■

If in local coord. $f: w = h(z), \; h \in \mathbb{C}\{z\}$ then $mult_p(f) = 1 + ord_p(\frac{dh(z)}{dz})$

Compact Riemann Surfaces of Genus g, g'

For holomorphic $f: C \to C'$

Cor: $\deg f := \sum_{p \in f^{-1}(q)} mult_p(f) \ \ \forall q \in C'$ (from Mustazee's talk)

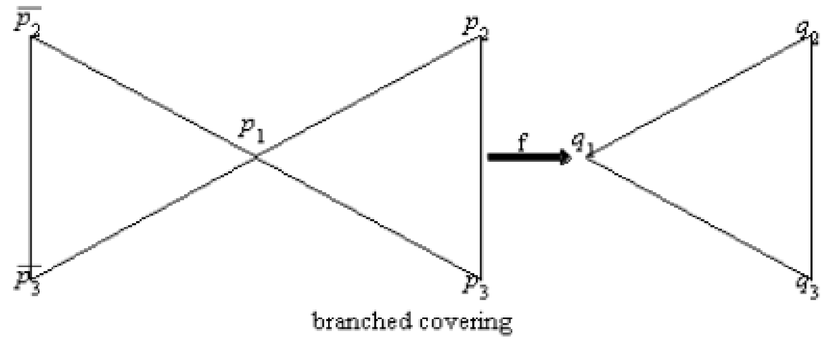
For compact C and C' there are finitely many branch/ramifiction points

Theorem (Hurwitz): $2g - 2 = \deg f\ (2g' - 2) + \sum_{p \in C} [mult_p(f) - 1]$.

Proof: Pick a triangulation of C' s.th. branch points are vertices

Proof of Hurwitz Formula

say v' vertices, e' edges, t' faces.



f a 'branched covering map' $\Longrightarrow$ triangulation lifts via f to C

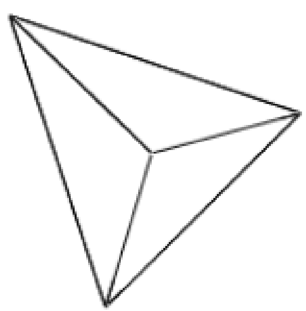
with v vertices, $e = \deg f \cdot e'$ edges and $t = \deg f \cdot t'$ faces.

$$|f^{-1}(q)| = \deg f + \sum_{p \in f^{-1}(q)} [1 - mult_p(f)] \quad \forall q \in C'$$

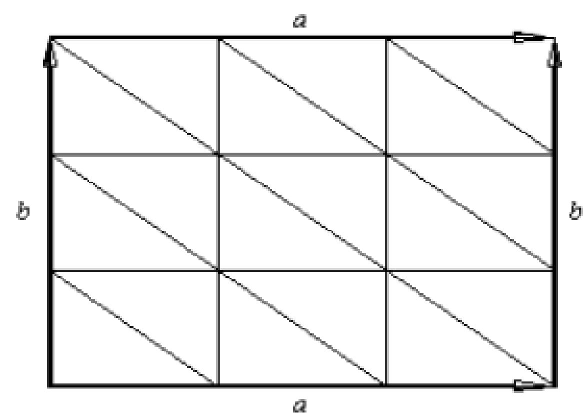
$$\Longrightarrow v = \sum_{q \text{ vertex of } C'} |f^{-1}(q)| = \deg f \cdot v' - \sum_{p \epsilon C} [mult_p(f) - 1]$$

$$2g - 2 = -\chi(C) \qquad \text{(for details see below)}$$

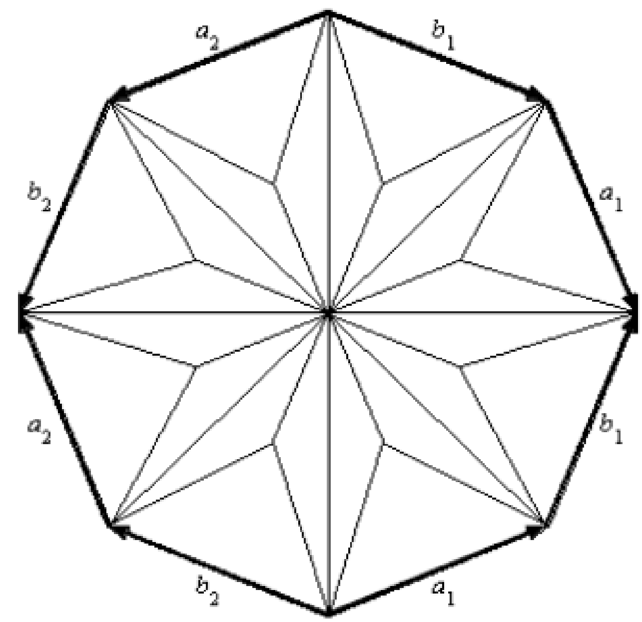
$$-\chi(C) = -\deg f\,(v' - e' + t') + \sum_{p\in C}\left[mult_p(f) - 1\right] \quad \blacksquare$$



$\underline{S^2: g = 0}$
$v = 4, e = 6, t = 4$
$\chi(S^2) = 2 = 2 - 2g$



$\underline{T^2: g = 1}$
$v = 9, e = 27, t = 18$
$\chi(T^2) = 0 = 2 - 2g$



$\underline{C\ (2 - handles): g = 2}$
$v = 10, e = 36, t = 24$
$\chi(C) = -2 = 2 - 2g$

Assignment: for $g > 2$ it follows that $\chi(C) = 2 - 2g$.

Proof of Plücker's Formula

C affine: $f(x,y)=0$ and let $E := \left\{\frac{\partial}{\partial y} f(x,y) = 0\right\}$. f irreducible $\Longrightarrow$

$(C \cdot E) := \sum_p mult_p(C \cap E) = d(d-1)$ (Bezout Thm on Slide 16).

<u>Exercise</u>: given finite # lines $\subset \mathbb{C}^3 \exists$ plane$\approx \mathbb{C}^2 \not\ni$ any of these lines

$\Longrightarrow$ can choose coordinates $s.th. C \cap L_\infty = d$ (Bezout Thm on Slide 16).

$\pi: \mathbb{CP}^2 \supset C \ni [x:y:z] \longmapsto [x:z] \in \mathbb{CP}^1 \quad \Rightarrow \deg \pi = d$.

<u>**Lemma**</u>: $p \in C_0/sing(C_0)$ then $\left.\frac{\partial f}{\partial y}\right|_p = 0 \Leftrightarrow p$ ramification point of π

<u>**Proof**</u>:" ... $\Longrightarrow \cdots$ " $\left.\frac{\partial f}{\partial x}\right|_p \neq 0$. IFThm $\Rightarrow$ exists $x = x(y)$ $s.th.$ $f(x(y),y) = 0$

$$0 = \frac{\partial}{\partial y} f(x(y), y) = \frac{\partial}{\partial x} f(x(y), y) x'(y) + \frac{\partial}{\partial y} f(x(y), y) \text{ near } p$$

centred at $(0,0) \Longrightarrow x'(0) = 0 \Leftrightarrow p$ ramification point of π.

Conversely if $\left.\frac{\partial f}{\partial y}\right|_p \neq 0$ then π provides a chart at p. Done.

$$(E \cdot C)_p \coloneqq mult_p(E \cap C) = ord_p\left(\frac{\partial}{\partial y} f(x(y), y)\right)$$ (see Slide 18)

$$= mult_p(\pi) - 1 \quad \text{then} \quad (E \cdot C) = \sum_{p \in ram(\pi)} (E \cdot C)_p = \sum_{p \in C} [mult_p(\pi) - 1]$$

Hurwitz Thm (Slide 7 with $f = \pi, \deg \pi = d, g' = 0) \Longrightarrow$

$$d(d-1) = 2g - 2 + 2d \Longrightarrow g = \frac{(d-1)(d-2)}{2}. \quad \blacksquare$$

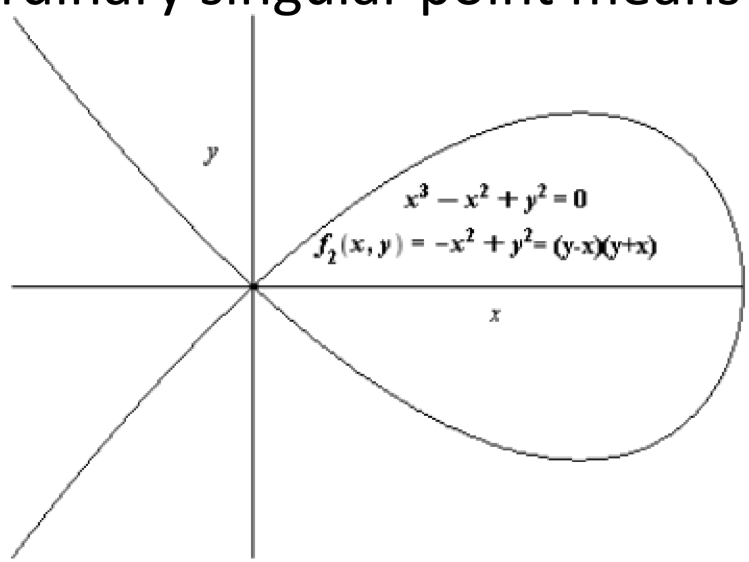
Types of Singular Points

$p \in C$ choose coord. on $\mathbb{CP}^2$ s.th. $p = [0:0:1] \Rightarrow f(0,0) = 0$.

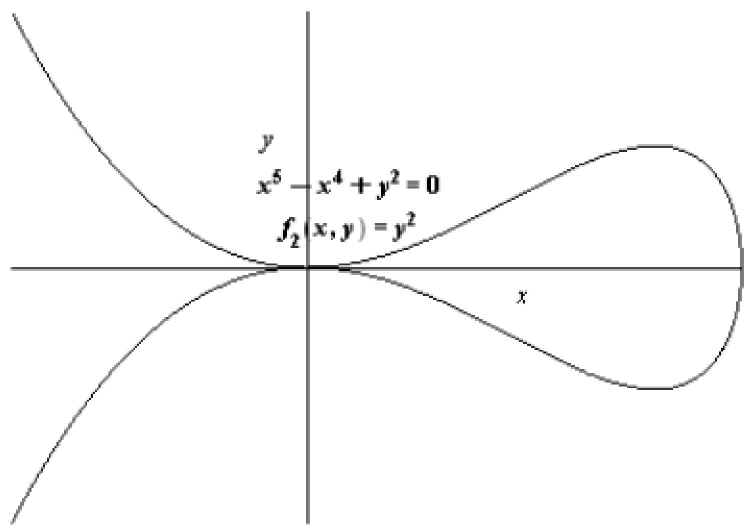
Then $f(x,y) = f_k(x,y) + \cdots + f_d(x,y)$, f_j homog, $\deg f_j = j$

$f_k \not\equiv 0, p \in sing(C) \Leftrightarrow mult_p f = k > 1$ (for $k = 1$: $f_1(x,y) = ax + by$)

Ordinary singular point means k distinct tangents at p.



Ordinary Double Point



Not Ordinary Double Point

Plücker with n ordinary double points

Replace π with $\psi = \pi \circ \sigma: \tilde{C} \to \mathbb{CP}^1$. Choose coordinates s.th.

1) $C \cap L_\infty = d$ 2) no vertical tangents in $sing(C)$.

$p \notin sing(C) \Rightarrow (E \cdot C)_p = mult_p(\psi) - 1$ (see Slide 11).

Say $\{p_j\}_{j=1}^{n}$ are double points ($k = 2$). In local coord centred at p_j:

$$f(x,y) = ax^2 + 2bxy + cy^2 + f_3(x,y) + \cdots + f_d(x,y)$$

$$\Longrightarrow ac - b^2 \neq 0 \;\; and \;\; \frac{\partial f(x,y)}{\partial y} = 2bx + 2cy + \cdots \quad \text{and}$$

2) $\Rightarrow c \neq 0$ at p_j. $\exists \;\; y(x) = -\frac{b}{c}x + \cdots$ s.th. $\frac{\partial f(x,y(x))}{\partial y} = 0$ (IFThm)

$$f\big(x, y(x)\big) = ax^2 + 2bxy(x) + cy(x)^2 + \cdots = \frac{ac-b^2}{c}x^2 + \cdots$$

As before $(C \cdot E)_{p_j} = ord_{p_j}\Big(f\big(x, y(x)\big)\Big) = 2$ (in total 2n)

$d(d-1) = (C \cdot E) = 2g - 2 + 2d + 2n$ (obtained using Slide7)

$$\Longrightarrow g = \frac{(d-1)(d-2)}{2} - n \quad \blacksquare$$

Proof of P irreducible $\Rightarrow \mathcal{D}_P\big(a(x)\big) \not\equiv 0$

Assume $\mathcal{D}_P \equiv 0, \mathbb{K} := \mathbb{C}(x) \Rightarrow \exists y_* \in \overline{\mathbb{K}}$ $s.th.$ $P(y_*) = P'(y_*) = 0.$

$\mathbb{K}[y]$ is a PID. $\exists \tilde{Q} \in \mathbb{K}[y]$ of min. degree, monic with $\tilde{Q}(y_*) = 0$

$\Rightarrow \deg \tilde{Q} \leq \deg P' < \deg P.$ $\mathbb{C}[x, y] \subset \mathbb{K}[y] \Rightarrow P \in \tilde{Q} \cdot \mathbb{K}[y] \Rightarrow$

$P(y, x) = \tilde{Q}(y, x)\tilde{G}(y, x)$. $\exists g, q \in \mathbb{C}[x]$ and $Q, G \in (\mathbb{C}[x])[y]$ s.th.

$Q = q\tilde{Q}, G = g\tilde{G}$ and $gcd = 1$ (in $\mathbb{C}[x]$ for coeff's of Q and of G)

$\Longrightarrow q(x)g(x)P(y, x) = Q(y, x)G(y, x) \Longrightarrow g, q \in \mathbb{C}$ (?!). Done.

Bezout Thm: Proof $(C \cdot E) = d(d-1)$

$F(x,y,z) = y^d + \cdots$ and $\mathcal{D}_F \not\equiv 0 \implies \exists$ a line in $\mathbb{C}^2$ $s.th. \mathcal{D}_F \not\equiv 0$

Pick coord (x,z) with $\{z=0\}$ being this line $\Rightarrow \mathcal{D}_F|_{z=0} \not\equiv 0.$

$\mathcal{L}f(x,y) \coloneqq F(x,y,0) = \prod_{1\leq j\leq d}(y-\lambda_j x)$. Choose y-axis with only single

points of $C \cap E$ on lines parallel to y-axis $\Rightarrow |C \cap E| = \left|\{\mathcal{D}_f = 0\}\right| \leq \deg_x \mathcal{D}_f$

$= \deg_x \mathcal{D}_{\mathcal{L}f}$ (use $\mathcal{D}_{\mathcal{L}f} = \mathcal{D}_F|_{z=0} \not\equiv 0$)

Say $$y^d + \sum_{j=1}^{d} a_j x^j y^{d-j} \coloneqq \mathcal{L}f(x,y)$$

$$\mathcal{P}^{p-1} \times \mathcal{P}^{q-1} \ni (F, G) \longmapsto F \cdot Q + G \cdot P \;\; with \;\; P(y, a(x)) \coloneqq \; \mathcal{L}f(x, y) \; and \;\; Q \coloneqq P'_y$$

$$Res_{P,Q} = \begin{pmatrix} d & 0 & \vdots & 0 & 1 & 0 & \vdots & 0 \\ (d-1)a_1x & d & \vdots & \vdots & a_1x & 1 & \vdots & \vdots \\ \vdots & (d-1)a_1x & \vdots & \vdots & \vdots & a_1x & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \vdots & a_1x \\ a_{d-1}x^{d-1} & \vdots & \vdots & d & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{d-1}x^{d-1} & \vdots & (d-1)a_1x & a_dx^d & \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots & 0 & a_dx^d & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & a_{d-1}x^{d-1} & 0 & 0 & \vdots & a_dx^d \end{pmatrix}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{d} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{d-1}$$

$\Rightarrow d(d-1) = \deg_x \mathcal{D}_{\mathcal{L}f}(x) \geq |C \cap E|$ (assignment for Joho… ■)

Hint: with $\Phi \coloneqq (F, F'_y, z) \;\Rightarrow\; \Phi|_{z=0} = (\mathcal{L}f, \mathcal{L}f'_y)$ and $\varphi \coloneqq (f, f'_y)$ are proper (using $\mu_0(\psi) = \dim_{\mathbb{C}} \mathcal{O}/(\psi) \;\Rightarrow d(d-1) \geq (C \cdot E) \coloneqq \deg \varphi = \deg \Phi$ $= \mu_0(\Phi) = \mu_0(\Phi|_{z=0}) \geq d(d-1)$ due to $\mathcal{O}_x/(\mathcal{D}_{\mathcal{L}f}) \hookrightarrow \mathcal{O}_{x,y}/(\mathcal{L}f, \mathcal{L}f'_y)$).

Proof $(C \cdot E)_p = ord_p(g(x(y), y)$

From Joho's talk: $\dim_{\mathbb{C}} \mathcal{O}_p/(g, f) = \mu_p(g, f) = |U_p \cap (f, g)^{-1}(b)|$,

for generic points b near $0 \in \mathbb{C}$ and U_p a 'small' neighbourhood of p.

If $df(p) \neq 0$ IFThm $\Rightarrow f \approx X = x - y(x)$ locally $\Longrightarrow$

$\mathcal{O}_p/(g, f) \cong \mathbb{C}\{y\}/g|_{\{X=0\}} \cong \mathcal{P}^{k-1}$ whenever

$$g\Big|_{\{X=0\}} \coloneqq g(x(y), y) = y^k(1 + \cdots)$$

$$\Longrightarrow \dim_{\mathbb{C}} \mathcal{O}_p/(g, f) = ord_p(g(x(y), y) \qquad \blacksquare$$