Contradiction Vs. Contraposition and Other Logical Matters

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In this document, the definitions of *implication*, *contrapositive*, *converse*, and *inverse* will be discussed and examples given from everyday English.

The statement "A implies B" can be written symbolically as "$A \to B$". This is the general form for an *implication*. For example, if A is the phrase "this figure is a triangle" and B is the phrase "this figure has three sides", then the symbols "$A \to B$" represents the sentence "If this figure is a triangle, this implies that it has three sides". Another way of interpreting the same set of symbols is: "If this figure is a triangle, then it has three sides."

The *contrapositive* of an the implication "A implies B" is "Not B implies not A", written "$\sim B \to \sim A$". These two statements are logically equivalent to one another. In other words, one is true if and only if the other is true. Referring to the above example, the contrapositive of "If this figure is a triangle, then it has three sides", is "If this figure does not have three sides, then it is not a triangle".

Another example of an implication and its contrapositive is:

> If it is raining, I will take my umbrella.
> I did not take my umbrella, therefore it did not rain.

Here, A represents the phrase "it is raining" and B represents "I take my umbrella". The second line is the contrapositive of the first and the first is the contrapositive of the second. If either is true, then so is the other. If either is false, then so is the other.

Phrases that mean the same as "A implies B ":

- If A is true then B is true.
- Whenever A is true, B is also true.
- Only if B is true, then A is true.

The *converse* of the implication "$A \rightarrow B$" is the implication "$B \rightarrow A$. As one can see, this is different than the contrapositive. Therefore, the "opposite" of an implication is not a well-defined concept. (*Note:* In mathematics, the description "well-defined" means to exist and have a single clear definition. In this case the opposite of an implication can have more than one possible meaning so is therefore not well-defined.)

The contrapositive of the converse is the *inverse*, written $\sim A \rightarrow\sim B$. To return to the umbrella example, the converse of the original implication is "If I take my umbrella, it implies that it is raining". The inverse becomes: "If it is not raining, then I must not have taken my umbrella".

A Note on Negation: If the statements A and B contain quantifiers, their negation will also contain quantifiers. For example, if A is the statement "All $x \in \mathbb{R}$ have a certain property". The negation of A, written $\sim A$ can be written as "It is not the case that all $x \in \mathbb{R}$ have that property" or "There exists an $x \in \mathbb{R}$ which does not have that property". In other words, the negation of a universal quantifier ($\forall$ = "for all") is an existential quantifier ($\exists$ = "there exists").

Likewise, the negation of the statement "There exists an $x \in \mathbb{R}$ that has a certain property" can be written as "It is not the case that there exists an $x \in \mathbb{R}$ such that it has this property" or "It is not the case that for all $x \in \mathbb{R}$, x has this property".

The phrase "It is not the case that all $x \ldots$" can be symbolized as "$\sim \forall x \ldots$". Whereas, the phrase "There does not exist an $x \ldots$" can be symbolized by "$\sim \exists x \ldots$". To state that there exists a unique x satisfying some property, we say "$\exists ! x \ldots$.