MAT 1000 / 457 : Real Analysis I
Assignment 1, due September 18, 2013

1. *(Folland 1.1)* A non-empty family of sets $\mathcal{R} \in \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences (i.e., if $E, F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$ and $E \setminus F \in \mathcal{R}$). A ring which is closed under countable unions is called a σ**-ring**.

   (a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

   (b) Let $\mathcal{R}$ be a ring. Then $\mathcal{R}$ is an algebra, if and only if $X \in \mathcal{R}$.

   (c) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

   (d) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}\}$ is a σ-algebra.

2. Consider the collection of subsets of $\mathbb{N}$ that have a well-defined density,

$$\mathcal{C} = \left\{ A \subset \mathbb{N} \;\middle|\; \lim_{n\to\infty} \frac{1}{n} \#(A \cap \{1, \dots, n\}) \quad \text{exists} \right\}.$$

   Is $\mathcal{C}$ a ring?

3. *(Folland 1.3)* Let $\mathcal{M}$ be an infinite σ-algebra. Show that ...

   (a) $\mathcal{M}$ contains an infinite sequence of disjoint non-empty sets;

   (b) $\mathcal{M}$ is uncountable.

4. *(Folland 1.4)* Let $\mathcal{A}$ be an algebra. Suppose that $\mathcal{A}$ is closed under countable increasing unions, i.e., $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ whenever $E_j \in \mathcal{A}$ and $E_j \subset E_{j+1}$ for each $j \in \mathbb{N}$.

   Prove that $\mathcal{A}$ is a σ-algebra.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) *(Inclusion-Exclusion, Folland 1.9)*
   If $E, F \in \mathcal{M}$, then $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$.

   (b) *(Restricting a measure to a subset, Folland 1.10)*
   Given a set $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Prove that μ_E is a measure.

6. *(Folland 1.8)* Let $(X, \mathcal{M}, \mu)$ be a measure space, and consider a sequence $(E_j)_{j\geq 1}$ in $\mathcal{M}$. Define

$$\liminf E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j \,, \qquad \limsup E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \,.$$

(a) Show that

$$\begin{aligned} \liminf E_j &= \{x \mid x \in E_j \text{ for all but finitely many } j\}\,, \\ \limsup E_j &= \{x \mid x \in E_j \text{ for infinitely many } j\}\,. \end{aligned}$$

Conclude that $\liminf E_j \subset \limsup E_j$.

(b) Give an example of a sequence (E_j) where $\liminf E_j \neq \limsup E_j$.

(c) Show that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.
If $\mu(\bigcup_{j=1}^{\infty} E_j) < \infty$, then also $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.