MAT 1000 / 457 : Real Analysis I
Assignment 10, due Friday December 6, 2013

1. *(Folland 3.7)* Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Show that, for every measurable set $E \subset X$,

   (a) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$, and correspondingly for ν^-;

   (b) the total variation measure $|\nu| = \nu^+ + \nu^-$ satisfies

$$|\nu|(E) = \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| : n \geq 0, E_1, \ldots, E_n \subset E \text{ disjoint} \right\}.$$

2. *Stein and Shakarchi, Exercise 3.4)* Let f be an integrable function on $\mathbb{R}^d$ with $||f||_{L^1} = 1$.

   (a) Show that its maximal function satisfies

$$Hf(x) \geq \frac{c}{|x|^d} \quad (|x| \geq 1)$$

   for some $c > 0$. Conclude that Hf is not integrable on $\mathbb{R}^d$.
   (*Hint:* Use that $\int_B |f| > 0$ for some ball B.)

   (b) Show that the weak-type estimate provided by the Hardy-Littlewood Maximal Theorem is best possible in the following sense: If f is supported in the unit ball, then

$$m(\{x : Hf(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

   for some $c' > 0$ and all sufficiently small $\alpha > 0$.

3. *(Folland 3.41)* Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval $I \subset [0, 1]$ of positive length (as constructed in Problem 6 of Assignment 5).

   (a) Let $F(x) = m([0, x] \cap A)$. Then F is absolutely continuous and strictly increasing, but F' vanishes on a set of positive measure.

   (b) Let $G(x) = m([0, x] \cap A) - m([0, x] \setminus A)$. Then G is absolutely continuous, but not monotone on any subinterval $I \subset [0, 1]$.

4. Let $\{f_n\}_{n\geq 1}$, f, g be functions in $L^2[0,1]$, with $f_n \to f$ pointwise a.e. If $|f_n(x)| < |x|^{-\frac{1}{3}}$, prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\,dx = \int_0^1 f(x)g(x)\,dx\,.$$

5. Let ϕ be a smooth function with compact support on $\mathbb{R}^d$ and $\int \phi = 1$.

(a) For $\delta > 0$, define $\phi_\delta(x) = \delta^{-d}\phi(x/\delta)$. Convince yourself that $\int \phi_\delta = 1$.

(b) If f is locally integrable, prove that

$$\lim_{\delta\to 0} \phi_\delta * f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d\,.$$

6. (a) Let U, V be non-empty open sets in $\mathbb{R}^n$, and let $T : U \to V$ be a bijection. Assume that for subsets $E \subset U$, the image $T(E)$ is (Lebesgue) measurable if and only if E is measurable. Prove that the **pushforward** of Lebesgue measure to V, given by

$$T\#m(F) = m(T^{-1}(F))\,, \quad \text{for } F \subset V$$

is absolutely continuous with respect to Lebesgue measure.
Hint: Use that sets of positive measure contain non-measurable subsets.

(b) If T is a diffeomorphism, find the density of $T\#m$ with respect to Lebesgue measure, i.e., find a function f such that $d(T\#m) = f dm$.