MAT 1000 / 457 : Real Analysis I
Assignment 2, due September 25, 2013

1. *(Dirac δ-measures on the real line)*
   Let $a \in \mathbb{R}$. The measure defined on $\mathcal{P}(\mathbb{R})$ by $\delta_a(E) = 1$ for $x \in E$, and $\delta_a(E) = 0$ otherwise, is called the **Dirac mass** at a.

   If $\mu : \mathcal{P}(\mathbb{R}) \to \{0, 1\}$ is a measure that takes only the values 0 and 1, prove that either $\mu = 0$, or else $\mu = \delta_a$ for some $a \in \mathbb{R}$.

2. *(Points of convergence)*
   Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative continuous functions on $\mathbb{R}$. Prove that

$$\left\{x \in \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} f_n(x) < \infty\right\}$$

   is a Borel set.

3. (a) *(Inclusion-exclusion)*
   Let $(X, \mathcal{M}, \mu)$ be a measure space. If $A_1, \ldots, A_n$ are sets of finite measure in $\mathcal{M}$, prove that

$$\mu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{\emptyset \neq F \subset \{1,\ldots,n\}} (-1)^{1+\#F} \mu\Big(\bigcap_{i\in F} A_i\Big) .$$

   (b) *(The forgetful secretary)*
   Let S_n be the set of all permutations of $\{1, \ldots, n\}$, and let μ_n be the normalized counting measure defined by $\mu_n(A) = \#A/n!$. Find

$$p_n = \mu_n(\{\pi \in S_n : \pi(i) \neq i \text{ for } i = 1, \ldots, n\}) ,$$

   and compute $p = \lim p_n$.

4. *(The section property of Borel sets)*
   Let A be a Borel set in $\mathbb{R}^2$. Prove that, for every $y \in \mathbb{R}$, the cross section

$$A(y) = \{x \in \mathbb{R} : (x, y) \in A\}$$

   is a Borel set.

5. *(Summable ⇒ countable)*
   Let Λ be an infinite set. For each $\lambda \in \Lambda$, let x_λ be a nonnegative number. Define the value of the series $\sum x_\lambda$ as the supremum of its finite partial sums,

$$\sum_{\lambda \in \Lambda} x_\lambda := \sup_{n \geq 0} \sup_{\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda} \{x_{\lambda_1} + \ldots + x_{\lambda_n}\} .$$

   (Note that the supremum is well-defined, even when its value is infinite.)

   If

$$\sum_{\lambda \in \Lambda} x_\lambda < \infty ,$$

   prove that

$$\Lambda' = \{\lambda \in \Lambda : x_\lambda > 0\}$$

   is countable.

6. *(Unions over chains of closed sets)*
   Let $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ be a collection of closed subsets of the real line, with the property that for each pair of sets $A, B \in \mathcal{C}$, either $A \subset B$ or $B \subset A$. Prove that $\bigcup_{A \in \mathcal{C}} A$ is a Borel set.