MAT 1000 / 457 : Real Analysis I
Assignment 3, due October 2, 2013

1. *(Stein & Shakarchi 1.11)* Let $A \subset [0,1]$ consist of all numbers whose decimal expansion does not contain the digit 4. Find $m(A)$.

2. *(Periodic sets)*
A set of integers $E \subset \mathbb{Z}$ is **periodic**, if $p + E = E$ for some natural number p, i.e.,
$$p + x \in E \iff x \in E\,.$$
In that case, we call p a **period** of E. If p is the minimal period of E, we denote by
$$\rho_0(E) = \frac{1}{p}\#(E \cap \{1, \dots, p\})$$
the **density** of E.
Prove that the periodic sets form an algebra, and that ρ_0 is a finitely additive measure.

3. *(Outer and inner regularity)*
Let μ^* be an outer measure on X induced from a premeasure μ_0, and let μ be the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets.
(a) *(Folland 1.18)* Prove that $\mu^*(A) = \inf_{E \in \mathcal{M}: E \supset A} \mu(E)$ for all $A \subset X$.
(b) *(Folland 1.19)* If $\mu_0(X) < \infty$, define the **inner measure** of a set $A \subset X$ by $\mu_*(A) = \mu_0(X) - \mu^*(A^c)$. Prove that A is measurable, if and only if $\mu^*(A) = \mu_*(A)$.

4. *(The Borel-Cantelli lemma)*
Let $(X, \mathcal{M}, \mu)$ be a measure space, let (E_j) be a sequence of measurable sets, and let $\limsup E_j$ be the set of points that lie in infinitely many of the E_j (see Assignment 1.4).
If
$$\sum_{j=1}^{\infty} \mu(E_j) < \infty\,,$$
prove that
$$\mu(\limsup E_j) = 0\,.$$

5. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of positive measure.

   (a) *(Folland 1.30)*
   Prove thay for every $\alpha < 1$ there is an open interval I such that $m(E \cap I) \geq \alpha m(I)$.

   (b) *(Folland 1.31)*
   Conclude that the set $E - E := \{x - y \,|\, x, y \in E\}$ contains an open interval centered at 0.

6. Let $N \subset \mathbb{R}$ be a set of Lebesgue measure zero. Prove that there exists $c \in \mathbb{R}$ such that the translated set $c + N := \{c + x \,|\, x \in N\}$ contains no rational point.