MAT 1000 / 457 : Real Analysis I
Assignment 4, due October 9, 2013

1. *(Folland 2.3)*
   If (f_n) is a sequence of measurable functions, then $\{x \mid \lim f_n(x) \text{ exists}\}$ is a measurable set.

2. *(Folland 2.14)*
   Let f be a nonnegative measurable function on a measure space $(X, \mathcal{M}, \mu)$.
   For $E \in \mathcal{M}$, set $\lambda(E) = \int_E f \, d\mu$. Show that ...

   (a) ... λ is a measure;

   (b) ... $\int g \, d\lambda = \int fg \, d\mu$ for every nonnegative measurable function g.

3. *(Folland 2.16)* If f is a nonnegative integrable function, then, for every $\varepsilon > 0$ there exists a set E of finite measure such that $\int_E f > (\int f) - \varepsilon$.

4. Let $x \in (0, 1)$, and let $(x_i)_{i \geq 1}$ be its decimal expansion.
   (If x has several decimal expansions, use the one that terminates in 0.)

   (a) Show that
   $$f(x) = \limsup_{n \to \infty} \left(\frac{1}{n} \# \{ i = 1, \ldots, n \mid x_i = 7 \} \right)$$
   defines a Borel measurable function on the unit interval.

   (b) Show that f assumes every value in $[0, 1]$ on each nonempty subinterval $(a, b) \subset (0, 1)$.

   (c) Construct a Borel measurable function that assumes every value in $[-\infty, \infty]$ on each nonempty subinterval of $(0, 1)$.

5. Let $(f_n)_{n \geq 1}$ be a sequence of measurable real-valued functions on $\mathbb{R}$. Prove that there exist constants $c_n > 0$ such that the series $\sum c_n f_n(x)$ converges for almost every $x \in \mathbb{R}$.
   (*Hint:* Borel-Cantelli.)

6. Please read the second half of Section 1.5 in Folland, on Lebesgue measure. Imitate the construction on p. 38 to produce a **fat Cantor set**: a totally disconnected, nowhere dense, compact subset C of the unit interval $[0, 1]$ that has positive Lebesgue measure. Argue that $m(C)$ can can be arbitrarily close to 1.
   Hint: Construct the set recursively, as an intersection of a decreasing chain $(C_i)_{i \geq 0}$ of compact sets. A useful fact is that such an intersection is always non-empty. You may also use without proof that for any sequence (γ_i) in $(0, 1)$,
   $$\prod_{i=1}^{\infty} (1 - \gamma_i) \text{ converges to a positive value} \quad \iff \quad \sum_{i=1}^{\infty} \gamma_i \text{ converges} .$$
   (To see this, take a logarithm and then note that $\lim_{\gamma \to 0} \gamma^{-1} \log(1 - \gamma) = -1$.)