MAT 1000 / 457 : Real Analysis I
Assignment 5, due October 16, 2013

1. *(Folland 2.17)*
   Assume Fatou's lemma and deduce the Monotone Convergence Theorem from it.

2. Let f be an integrable function with the property that $\int_E f \geq 0$ for every measurable set E. Prove that f is nonnegative almost everywhere. In particular, if $||f||_{L^1} = 0$ then $f = 0$ a.e.

3. *(The Nikodym distance)* Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

   (a) Verify that $d(A, B) = \mu(A \triangle B)$ defines a metric on $\mathcal{M}/\!\sim$ with a suitable equivalence relation $\sim$. Here,

   $$A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

   is the **symmetric difference** of A and B. (Be brief!)

   (b) Prove that the metric space is complete.

   Remark: Please work directly with the measures, without referring to L^1 or the Dominated Convergence Theorem.

4. *(Stein & Shakarchi 1.5)* Given a non-empty set $E \subset \mathbb{R}^n$, consider the open sets

   $$U_n = \left\{ x \in \mathbb{R}^n : d(x, E) < \tfrac{1}{n} \right\} .$$

   (a) If E is compact, prove that $m(E) = \lim m(U_n)$.

   (b) Give an example of a bounded open set where the conclusion fails.

5. *(Composition does not respect Lebesgue measurability; Folland 2.9)*
   Let f be the Cantor-Lebesgue function (the 'devil's staircase') from Section 1.5, and let $g : [0, 1] \to [0, 2]$ be defined by $g(x) = f(x) + x$. Prove the following assertions.

   (a) g is bijective, and $h = g^{-1}$ is continuous.

   (b) If C is the Cantor set, then $m(g(C)) = 1$.

   (c) Let $A \subset g(C)$ be a nonmeasurable set. Then $B := g^{-1}(A)$ is a Lebesgue measurable set. Hence $\mathcal{X}_A = \mathcal{X}_B \circ h$ is not a Lebesgue measurable function.

   Remark: You may take for granted that every set of positive Lebesgue measure contains a nonmeasurable subset (see Folland Problem 1.29).

6. *(Folland 1.33)* Construct a Borel set A such that $0 < m(A \cap I) < m(I)$ for every non-empty open subinterval $I \subset [0, 1]$.