MAT 1000 / 457 : Real Analysis I
Assignment 6, due October 23, 2013

1. *(Folland 2.28d)* For $a \in \mathbb{R}$, compute
$$\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\,dx\,,$$
and justify the calculations. (*Remark:* The answer depends on the sign of a. How does it accord with the various convergence theorems?)

2. *(Difference term in Fatou's lemma)*
Let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions that converges pointwise a.e. to f.
(a) Prove that
$$\lim_{n\to\infty} \left\{ \int |f_n| - \int |f - f_n| \right\} = \int |f(x)|\,.$$
(b) Argue that this strengthens the conclusion of Fatou's lemma for $\liminf \int |f_n|$.

3. *(Folland 2.48)* Let $\mu = \nu$ be counting measure on $\mathbb{N}$. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. Find the values of $\iint f\,d\mu d\nu$, $\iint f\,d\nu d\mu$, and $\int |f|\,d(\mu\times\nu) = \infty$. (Why are they defined?)

4. *(Folland 2.59: The Dirichlet integral)* Show that
$$\lim_{b\to\infty} \int_0^b x^{-1}\sin x\,dx = \frac{\pi}{2}\,.$$
Hint: Integrate the function $e^{-xy}\sin x$ with respect to x and y, and use that
$$\int e^{-xy}\sin x\,dx = -e^{-xy}\left(\tfrac{1}{1+y^2}\cos x + \tfrac{y}{1+y^2}\sin x\right).$$
Please be careful ... the function $f(x) = x^{-1}\sin x$ is not integrable over $(0,\infty)$!

5. *(The Riemann-Lebesgue lemma)* If f is integrable on $\mathbb{R}$, prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)e^{-inx}\,dx = 0\,.$$

6. Does there exist a dense subset of $\mathbb{R}^2$ such that no three points are collinear?