MAT 1000 / 457 : Real Analysis I
Assignment 7, due November 13, 2013

1. (a) *(Parallelogram identity)* Let $\langle \cdot, \cdot \rangle$ be an inner product on a complex vector space V, and consider the norm defined by $||f|| = \sqrt{\langle f, f \rangle}$. Prove that

$$||f+g||^2 + ||f-g||^2 = 2\left(||f||^2 + ||g||^2\right) .$$

(b) Show that the parallelogram identity fails in $L^1(\mathbb{R})$. Hence the L^1-norm does not come from an inner product.

2. (a) Let (f_n) be a sequence of integrable functions that converges in L^1 to some limiting function f,

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0 \, .$$

Prove that there is a subsequence (f_{n_j}) that converges pointwise almost everywhere to f.

(b) *('Typewriter' sequences)* Construct an example of a nonnegative sequence of integrable functions (f_n) on the unit interval that converges to zero in L^1 such that the sequence of values $(f_n(x))$ converges for no $x \in (0, 1)$.

3. Let $(f_n)_{n\geq 1}$ be a sequence of functions on $[0, 1]$ that is bounded in L^2 (i.e., $\sup_n ||f_n||_2 < \infty$). Assume that there exists a measurable function f such that

$$\lim_{n\to\infty} \int_0^1 |f_n - f| \, dm = 0 \qquad (n \to \infty) \, .$$

Show that $f \in L^2$. Does it follow that $f_n \to f$ in L^2 ?

4. *(Lusin's theorem)* Let $E \subset \mathbb{R}^d$ be a set of finite measure, and let f be a measurable real-valued (or complex-valued) function on E. Given $\varepsilon > 0$, prove that there exists a compact set $K \subset E$ with $m(E \setminus K) < \varepsilon$ such that the restriction $f\big|_K$ is continuous.

Hint: Approximate f with a sequence of continuous functions and apply Egoroff's theorem.

5. *(Folland 2.60: The Beta-integral)*

   The Gamma-function is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ for $x > 0$. Show that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \qquad (x, y > 0)\,.$$

   Hint: Write $\Gamma(x)\Gamma(y)$ as double integral and change variables in the inner integral to simplify the exponential.

6. For $1 \le k \le n$, compute the spherical integral

$$\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} (u_1^2 + \cdots + u_k^2)^{-1/2}\, d\sigma(u)\,,$$

   where σ is the standard rotationally invariant surface measure on the unit sphere $\mathbb{S}^{n-1}$.

   Hint: Rewrite this as a Gaussian integral over $\mathbb{R}^n$. Write your answer either in terms of the Gamma-function or in terms of the measures ω_d of the d-dimensional unit balls.