MAT 1000 / 457 : Real Analysis I
Assignment 8, due November 20, 2013

1. *(Uniform convergence of Fourier series)* Let f be a continuously differentiable 2π-periodic function, and let $(a_k)_{k\in\mathbb{Z}}$ be the sequence of its Fourier coefficients.

   (a) Show that the series $\sum_{k\in\mathbb{Z}} k^2|a_k|^2$ converges.

   (b) Use Schwarz' inequality to verify that the sequence of partial sums (S_n), given by

   $$S_n = \sum_{k=-n}^{n} a_k e^{-ikx}$$

   satisfies the Cauchy criterion with respect to the supremum norm, $||g||_{\sup} = \sup_x |g(x)|$. Hence the Fourier series converges uniformly to f.

2. Find the values of

   $$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{1}{k^2}\,, \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{1}{k^4}\,, \qquad \text{(c)} \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\,.$$

   Hint: Compute the Fourier series of the function $f(x) = x$ for $x \in (-\pi, \pi)$.

3. *(The Dirichlet kernel)* For $n \in \mathbb{N}$, let P_n be the projection in $L^2(0, 2\pi)$ defined by

   $$P_n f(x) = \sum_{k=-n}^{n} a_k e^{ikx}\,.$$

   Here, (a_k) is the sequence of Fourier coefficients of f. Show that

   $$P_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-y) f(y)\, dy\,,$$

   where the integral kernel is given by

   $$D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}\,.$$

4. Let $(c_k)_{k\in\mathbb{Z}}$ be a bi-infinite sequence of complex numbers that is square summable,

   $$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty\,.$$

   Prove that (c_k) is the sequence of Fourier coefficients of some 2π-periodic function $f \in L^2$.

5. *(Fractional integrals, Folland 2.61)* If f is continuous on $[0, \infty)$, for $\alpha > 0$ and $x \geq 0$ let

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt\,.$$

   (a) Prove that $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$.
   Hint: Use Problem 6 from Assignment 7 / Folland 2.60.

   (b) If $n \in \mathbb{N}$, then $I_n f$ is an n-th order antiderivative of f.

6. *(Intermediate values for Lebesgue measure)* Let $A \subset B$ be compact sets in $\mathbb{R}^d$, and fix t with $m(A) < t < m(B)$. Prove that there exists a compact set K with $A \subset K \subset B$ and $m(K) = t$.