MAT 1000 / 457 : Real Analysis I
Assignment 9, due November 27, 2013

1. *(Folland 3.13)* Consider the unit interval $X = [0, 1]$ equipped with the Borel σ-algebra. Let m = Lebesgue measure, and μ = counting measure. Prove that

   (a) $m \ll \nu$, but $dm \neq f\, d\mu$ for any function f;

   (b) μ has no Lebesgue decomposition with respect to m.

   Why does that not contradict the Lebesgue-Radon-Nikodym theorem?

2. Let $\{f_n\}_{n\geq 1}$, f, g be functions in $L^2[0, 2\pi]$, with $f_n \to f$ pointwise a.e. If $||f_n||_{L^2} \leq M$ for all n and g is bounded, then

$$\lim_{n\to\infty} \int_0^{2\pi} f_n(x)g(x)\, dx = \int_0^{2\pi} f(x)g(x)\, dx\,.$$

3. As in Problem 4 of Assignment 4, let $(x_n)_{n\geq 1}$ be the decimal expansion of $x \in (0, 1)$. (If the expansion is non-unique, take the one that terminates in 0.) You will show that

$$\lim_{n\to\infty} \left(\frac{1}{n} \#\{i = 1, \dots, n : x_i = 7\} \right) = 0.1$$

   for almost every $x \in (0, 1)$.

   (a) Let $y_n(x) = \mathcal{X}_{\{x_n=7\}} - 0.1$ and $S_n(x) = \sum_{k=1}^n y_n(x)$. Check that

$$\int_{(0,1)} y_n = 0\,, \quad \int_{(0,1)} y_m y_n = 0 \quad \text{for } m \neq n\,, \quad \text{and} \int_{(0,1)} y_n^2 \leq 1\,.$$

   Use this to estimate $\int S_n^4$.

   (b) Show that

$$\int_{(0,1)} \sum_{n=1}^{\infty} \left(\frac{S_n(x)}{n} \right)^4 < \infty\,,$$

   and conclude that $S_n(x)/n \to 0$ for almost every x.

4. *(Kolmogorov's criterion)* Let $(\Omega, \mathcal{M}, \mu)$ be a probability space. A sequence of random variables $X_i : \Omega \to \mathbb{R}$, for $i = 1, 2, \ldots$ is called **independent**, if for every $N > 0$ and every $t_1, \ldots, t_N \in \mathbb{R}$,

$$P(X_1 > t_1, \ldots, X_n > t_n) = \prod_{i=1}^{N} P(X_i > t_i)\,.$$

If $(X_i)_{i \geq 1}$ is a sequence of independent random variables with $E(X_i) = 0$ for all i and

$$\sum_{i=1}^{\infty} E(X_i^2) < \infty\,,$$

prove that

$$P\left(\sum_{i=1}^{\infty} X_i \text{ converges}\right) = 1\,.$$

5. *(Convolution with a smooth kernel)* Let ϕ be a smooth complex-valued function $\mathbb{R}^d$ with compact support (i.e., ϕ vanishes outside some compact set $K \subset \mathbb{R}^d$.) If f is integrable, prove that the **convolution**

$$f * \phi(x) = \int f(x-y)\phi(y)\,dy$$

is smooth. Moreover,

$$\lim_{|x|\to\infty} f * \phi(x) = 0\,.$$

6. *(Lieb & Loss Problem 2.10)*
(a) Let f be a measurable real-valued function on the real line that is *additive* i.e.,

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}\,.$$

Prove that there exists an $\alpha \in \mathbb{R}$ such that $f(x) = \alpha x$, i.e., f is *linear*.

(b) Give an example of a (non-measurable) function that is additive but not linear.