UNIVERSITY OF TORONTO

The Faculty of Arts and Science

DECEMBER 2012 EXAMINATIONS
MAT457HIF

Duration – 3 hours

NO AIDS ALLOWED

Each problem is worth 20 points. Please be brief but justify your answers, citing relevant theorems. Sometimes a sketch can help!

1. Let (f_n) be a sequence of integrable functions with $\int |f_n| \leq n^{-2}$. Prove that f_n converges to zero pointwise almost everywhere.

2. Let $E \subset F$ be two compact sets in $\mathbb{R}^n$ with $E \subset F$. If $m(E) < c < m(F)$, find a compact set K with $E \subset K \subset F$ such that $m(K) = c$. (*Hint:* Use the geometry of $\mathbb{R}^n$ to construct a family of compact sets $K_t \subset \mathbb{R}^n$ whose measure depends continuously on t.)

3. Please ...

   (a) define *outer measure* and state Carathéodory's extension theorem;

   (b) define what it means for a function $F : \mathbb{R} \to \mathbb{R}$ to be *absolutely continuous*, and state the Fundamental Theorem of Calculus for Lebesgue integrals.

4. Let f, g be integrable functions on $\mathbb{R}^n$.

   (a) Prove that
   $$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy$$
   exists for almost every x, and defines an integrable function on $\mathbb{R}^n$.

   (b) Assume, additionally, that g is non negative, smooth, and compactly supported with $\int g = 1$, and set $g_\delta(x) = \delta^{-n} g(x/\delta)$. Prove that
   $$\lim_{\delta \to 0^+} f * g_\delta(x) = f(x) \quad \text{almost everywhere.}$$
   Hint: First show that
   $$|f * g_\delta(x) - f(x)| \leq \int |f(x-y) - f(x)|\, g_\delta(y)\, dy\,.$$

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5. (a) Define, in simple terms: What does it mean for a set $N \subset \mathbb{R}$ to have *measure zero*? What does it mean for a set $M \subset \mathbb{R}$ to be *meager*?

   (b) Write $\mathbb{R}$ as the disjoint union of a meager set and a null set, i.e, find M and N such that

$$M \cup N = \mathbb{R}\,, \qquad M \cap N = \emptyset\,,$$

   where M is meager and N has measure zero. (*Hint:* Use Cantor-like sets.)

6. Let f and g be two non negative integrable functions on a measure space $(X, \mathcal{M}, \mu)$. Prove that

$$||f - g||_{L^1} = \int_0^\infty \mu\big(\{x : f(x) > t\} \triangle \{x : g(x) > t\}\big)\, dt\,.$$

   Hint: Consider separately the sets $\{x : f(x) > g(x)\}$ and $\{x : g(x) > f(x)\}$.

— *end* —