MAT 1000 / 457 : Real Analysis I
Midterm Test, November 2, 2011

(Four problems; 20 points each. Time: 2 hours.)

Please be brief but justify your answers. Sometimes a sketch can help!

1. (a) Let Σ be a σ-algebra on a space X, and let μ be a measure on Σ. Consider the collection of subsets of X that differ from sets in Σ by a null set,

$$\Sigma' = \{A \cup N \mid A \in \Sigma,\ N \subset A_0 \in \Sigma,\ \mu(A_0) = 0\}\,.$$

   Prove that Σ' is a σ-algebra.

   (b) If Σ is the Borel σ-algebra on $\mathbb{R}^d$, what is Σ'?

   (c) Give an example of a null set in $\mathbb{R}^d$ that is not a Borel set.

2. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable real-valued functions. Prove that there exists a sequence of positive numbers c_n such that $\sum_{n=1}^{\infty} c_n f_n$ converges for almost every $x \in \mathbb{R}$.

3. Let $\{f_n\}_{n\geq 1}$ be a sequence of integrable functions that converges pointwise a.e. to an integrable function f.

   (a) Prove that

$$\lim_{n\to\infty} \left\{ \int |f_n|\, dm(x) - \int |f - f_n|\, dm(x) \right\} = \int |f(x)|\, dm(x)\,.$$

   (b) Show that this implies Fatou's lemma.

   (c) Give an example of a sequence of integrable functions on $[0, 1]$ where the inequality in Fatou's lemma is strict.

4. Let $f(x) = x^{-1} \sin x$.

   (a) Is f integrable over finite intervals $(0, b)$? Is it integrable over $(0, \infty)$? Why / why not?

   (b) Show that

$$\lim_{b\to\infty} \int_0^b f(x)\, dx = \frac{\pi}{2}\,.$$

   Hint: Integrate the function $e^{-xy} \sin x$ with respect to x and y, and use that

$$\int e^{-xy} \sin x\, dx = -e^{-xy} \left(\tfrac{1}{1+y^2} \cos x + \tfrac{y}{1+y^2} \sin x \right).$$

   Be careful ...