Practice Problems (collected at UVa)

1. Define ...

   measure, outer measure, σ-algebra, complete measure, σ-finite measure; product σ-algebra and product measure; Borel set, F_σ and G_δ sets, Lebesgue measurable set, outer regularity; measurable functions and integrable functions; Banach space, the spaces L^1 and L^2; inner product, orthogonality, norm, convergence in L^2; completeness; simple and really simple functions on $\mathbb{R}^d$; Fourier series and Fourier coefficients of a function; Poisson kernel and Dirichlet kernel; maximal function, density point of a set, Lebesgue point of a function, signed measures; Hahn decomposition and Jordan decomposition; absolutely continuous and mutually singular measures, Lebesgue-Stieltjes measures on $\mathbb{R}$; BV functions and absolutely continuous functions, total variation.

2. State ...

   continuity from above and below, the monotone class theorem, the Vitali-Hahn-Saks theorem, the Borel-Cantelli lemma, Carathéodory's extension theorem, the great convergence theorems; completeness of L^1; Egorov's theorem and Lusin's theorem; the theorems of Fubini and Tonelli; Kolmogorov's extension theorem; the change of variables formula; translation and rotation invariance of Lebesgue measure, Schwarz' inequality, Bessel's inequality and Parseval's identity, the Hardy-Littlewood maximal function theorem, Lebesgue's differentiation theorem in $\mathbb{R}^d$, Vitali's covering lemma; the Lebesgue-Radon-Nikodym theorem on a general measure space, on $\mathbb{R}^d$, and on $\mathbb{R}^1$; the Fundamental Theorem of Calculus.

3. True or False?

   (a) If a real-valued function on $(0,1)$ is differentiable, then it is measurable.

   (b) Every set of finite Lebesgue measure can be partitioned into two subsets of equal measure.

   (c) A subset of full Lebesgue measure in $(0,1)$ is necessarily dense.

   (d) If $E \subset \mathbb{R}^2$ is a measurable set with the property that its vertical cross sections $E_x = \{y \in \mathbb{R} \mid (x,y) \in E\}$ have measure zero for almost every x, then also the horizontal cross sections $E^y = \{x \in \mathbb{R} \mid (x,y) \in E\}$ have measure zero for a.e. y.

4. Let (f_n) be a sequence of continuous real-valued functions on $\mathbb{R}$, and fix $a \in \mathbb{R}$. Prove that

   $$A = \{x \in \mathbb{R} \mid \liminf f_n(x) < a\}$$

   is a Borel set.

5. Prove the Riemann-Lebesgue lemma: If f is integrable on $\mathbb{R}$, then

   $$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)e^{-inx}\,dx = 0\,.$$

6. Let A be a positive definite, symmetric $d \times d$ matrix. Compute the Gaussian integrals

   $$\int_{\mathbb{R}^d} e^{-x\cdot Ax}\,dm(x) \qquad \text{and} \qquad \int_{\mathbb{R}^d} |x|^2 e^{-x\cdot Ax}\,dm(x)\,.$$

7. (a) Is $\sum_{|n|\geq 1} n^{-\frac{3}{4}} e^{-nx}$ the Fourier series of a function in $L^2[0, 2\pi]$?

   (b) Is $\sum_{n\geq 0} \sin(n) e^{inx}$ the Fourier series of a function in $L^2[0, 2\pi]$?

   (c) Is $\sum_{n\geq 0} \sin(n) e^{inx}$ the Fourier series of a function in $L^1[0, 2\pi]$?

8. Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, and let $\mathcal{F}$ be a σ-algebra contained in Σ.

   (a) If f is a μ-integrable function on X, prove that there exists a $\mathcal{F}$-measurable function h with

$$\int_E f\, d\mu = \int_E h\, d\mu$$

   for all $E \in \mathcal{F}$. (*Hint:* Consider the measure ν obtained by restricting μ to $\mathcal{F}$.)

   (b) Find the function h if $X = (0, 1)$ with Lebesgue measure, and $\mathcal{F}$ is the σ-algebra generated by the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and $f(x) = x^2$.

9. Let $K(x, y)$ be a measurable complex-valued function on the unit square $0 \leq x, y \leq 1$ with

$$\int_0^1 \int_0^2 |K(x, y)|^2\, dxdy < \infty\,.$$

   Prove that if $f \in L^2[0, 1]$, then the integral

$$Tf(x) = \int_0^1 K(x, y) f(x)\, dy$$

   converges for a.e. x.

10. Is the function defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} e^{inx}\,.$$

   continuous? Differentiable? Evaluate $\int_0^{2\pi} |f(x)|^2\, dx$.

11. Let $\{f_n\}$ be a sequence of measurable real-valued functions on $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \int |f_n(x)|\, dx < \infty\,.$$

   Prove that the series $\sum f_n(x)$ converges for a.e. $x \in \mathbb{R}$, and that

$$\int \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \left(\int f_n(x)\, dx \right)\,.$$

12. Let $X = [0, 1]$ with Lebesgue measure m, and let $Y = [0, 1]$ with counting measure ν. If f is the characteristic function of the diagonal $\{(x, x) : x \in [0, 1] \subset X \times Y$. Show by evaluating both sides that

$$\int_X \left\{ \int_Y f d\nu \right\} dm \neq \int_Y \left\{ \int_X f dm \right\} d\nu\,.$$

   Why does this not contradict Fubini's theorem?

13. Let $\{f_n\}$ be a sequence of functions in $L^2[0,1]$ with $||f_n||_{L^2} \leq M$. Assume furthermore that there exists a measurable function f such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)| \, dm(x) = 0 \, .$$

Show that $f \in L^2$. Does it follow that f_n converges to f in L^2?

14. *(Folland 6.38)* Let f be a nonnegative measurable function on a measure space $(X, \mathcal{M}, \mu)$. Prove that

$$f \in L^1 \iff \sum_{k=-\infty}^{\infty} 2^k \mu(\{x : f(x) > 2^k\}) < \infty \, .$$

15. Suppose that μ, ν are Borel measures on $\mathbb{R}$ that agree on each interval $I \subset \mathbb{R}$. Prove that $\mu = \nu$.

16. Let f be a measurable function on $[0, \infty)$, and define

$$F(s) = \int_0^\infty \frac{f(x)}{(1+sx)^2} \, dx \, .$$

(a) If $\frac{f(x)}{x}$ is integrable prove that $F(s)$ is finite a.e., and that F is integrable over $[0, \infty)$.

(b) If $f(x) \geq 0$ and $F(s)$ is bounded, then f itself must be integrable.

(c) Assume that f is continuous, and that $a := \lim_{x\to\infty} f(x)$ exists. Find

$$\lim_{s\to 0} sF(s) \, , \quad \lim_{s\to\infty} sF(s) \, .$$

17. Let $f(x,t)$ be a real-valued function on $\mathbb{R}^2$ such that $f(\cdot, t)$ is continuous for every $t \in \mathbb{R}$. Suppose there exists an integrable function g such that

$$|f(x,t)| \leq g(t) \, , \quad \text{for all } x, t \in \mathbb{R} \, .$$

Prove that

$$F(x) = \int_{\mathbb{R}} f(x,t) \cos t \, dt$$

is bounded and continuous.

18. State two simple simple (useful) conditions, each of which guarantees that

$$\sum_{n=1}^{\infty} \left(\int f_n \, d\mu \right) = \int \left(\sum_{n=1}^{\infty} f_n \right) \, d\mu \, .$$

19. *(Folland 2.13)* Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions. Assume that $f_n \to f$ pointwise a.e., and that $\int f_n \to \int f$.

(a) If f is integrable, show that $\int_E f = \lim \int_E f_n$ for all measurable sets E.

(b) However, this need not be true if $\int f = \infty$.