Practice Problems (collected at UVa)

1. Define ...

   measure, outer measure, σ-algebra, complete measure, σ-finite measure; product σ-algebra and product measure; Borel set, F_σ and G_δ sets, Lebesgue measurable set, outer regularity; integrable function; Banach space, the space L^1; simple and really simple functions on $\mathbb{R}^d$

2. State ...

   continuity from above and below, the monotone class theorem, the Vitali-Hahn-Saks theorem, the Borel-Cantelli lemma, Carathéodory's extension theorem, the great convergence theorems; completeness of L^1; the theorems of Fubini and Tonelli; the change of variables formula; translation and rotation invariance of Lebesgue measure

3. Give an example of ...

   (a) a closed set of positive measure that has no interior;

   (b) a sequence of functions $\{f_n\}$ in $L^1(\mathbb{R})$ converging to zero pointwise a.e. but such that

   $$\lim \int_{\mathbb{R}} f_n(x)\, dx \neq 0\,;$$

   (c) a L^1-Cauchy sequence of functions which does not converge pointwise anywhere.

4. True or False?

   (a) If f is a measurable function on $[0,1]$, then the set

   $$C = \{x \in [0,1] : f \text{ is continuous at } x$$

   is measurable.

   (b) A subset of full Lebesgue measure in $(0,1)$ is necessarily dense.

   (c) A nowhere dense subset of $(0,1)$ has measure zero.

5. Assume that $E \subset \mathbb{R}$ has Lebesgue measure zero. Can the set

   $$G = \{(x,y) \in \mathbb{R}^2 : x - y \in E\}$$

   have positive Lebesgue measure?

6. Let f be a nonnegative measurable function on $\mathbb{R}^d$ with

   $$m\left(\{x : f(x) > \lambda\}\right) = \frac{1}{1+\lambda^2}\,.$$

   Compute the L^1-norm of f.

7. If f is an integrable function on $\mathbb{R}^d$ such that

$$\int_E f(x)\, dx = 0$$

for every measurable set E, prove that $f = 0$ a.e.

8. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $[0,1]$ with

$$\sum_{n=1}^{\infty} \int_0^1 f_n(x)\, dm(x) < \infty\,.$$

Show that except for x in a set of measure zero, $f_n(x) \geq 1$ occurs only for finitely many n.

9. Let $\{f_n\}_{n\geq 1}$ and f be real-valued measurable functions on $\mathbb{R}$.

(a) If $f_n \to f$ a.e., show that for any positive number $\varepsilon > 0$,

$$\lim_{n\to\infty} m\left(\{x : |f(x) - f_n(x)| > \varepsilon\}\right) = 0\,.$$

(b) What can you say about the converse?

10. If $\{f_n\}$ is a *fast* Cauchy sequence in $L^1(\mathbb{R}^d)$, in the sense that $||f_n - f_{n-1}||_{L^1} \leq 2^{-n}$, prove that $\lim f_n(x)$ exists for almost every x.

11. For $c > 1$, find

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-cx}\, dx\,.$$

12. Let $f(\lambda, x)$ be a continuous function of two variables on the unit square $0 < \lambda, x < 1$. Suppose that the partial derivative $\frac{\partial f}{\partial \lambda}(\lambda, x)$ exists for all λ and x, and that

$$h(x) = \sup_{0<\lambda<1} \left|\frac{\partial f}{\partial \lambda}(\lambda, x)\right|$$

is integrable. Show that the function $F(\lambda) = \int_0^1 f(\lambda, x)\, dx$ is differentiable and satisfies

$$F'(\lambda) = \int_0^1 \frac{\partial f}{\partial \lambda}(\lambda, x)\, dx\,.$$

13. Let f be a integrable function on $[0,1]$, and consider $S = \{x \in [0,1] : f(x) \text{ is an integer}\}$. Evaluate

$$\lim_{n\to\infty} \int_0^1 |\cos \pi f(x)|^n\, dx\,.$$

14. Evaluate $\lim_{\varepsilon \to 0^+} \int_0^\infty \frac{e^{-x}}{1+\varepsilon^2 x}\, dx.$

15. For which positive real numbers p does the integral

$$\int_0^1 \int_0^1 \frac{1}{(x^2+y^2)^p}\, dx dy$$

converge?

16. Let $Q = [0,1] \times [0,1]$ be the unit square. Show that

$$\int_Q \frac{1}{1-xy}\, dm = \sum_{n \geq 1} \frac{1}{n^2}\,.$$

17. Let f be an integrable function such that

$$\int_0^\infty x\, |f(x)|\, dx < \infty\,.$$

Prove that

$$\frac{d}{dt} \int_0^\infty \sin(xt)\, f(x)\, dx = \int_0^\infty x\, \cos(xt)\, f(x)\, dx\,.$$

18. Prove that

$$\lim_{n \to \infty} n \int_0^\infty e^{-x^2}(e^{\frac{x}{n}} - 1)\, dx = \int_0^\infty x e^{-x^2}\, dx = \frac{1}{2}\,.$$

19. Evaluate

$$\lim_{n \to \infty} \int_{\mathbb{R}} (1 - e^{-\frac{t^2}{n}}) e^{-|t|} \cos t\, dt\,.$$

20. Let M be a positive definite symmetric $n \times n$ matrix. Find the measure of the ellipsoid

$$E = \{x \in \mathbb{R}^n : x \cdot Mx < 1\}$$

in terms of M and the measure of the unit ball. (*Hint:* Diagonalize M.)

21. A function $f : (0, \infty) \to \mathbb{R}$ is *improperly Riemann integrable* if the Riemann integral

$$I(t) = \int_0^t f(x)\, dx$$

exists for every $t > 0$ and converges to some finite limit I as $t \to \infty$. If both f and $|f|$ are improperly Riemann integrable, prove that f is Lebesgue integrable and

$$I = \int_{(0,\infty)} f\, dm\,.$$