MAT 1600 : Probability I
Assignment 1, due September 21, 2016

1. (a) *Inclusion-exclusion*
   Show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any pair of events A, B.

   (b) *Conditional probability*
   Given an event E with $P(E) \neq 0$, define
   $$P(A|E) = \frac{P(A \cap E)}{P(E)}\,.$$
   Show that $P(\cdot|E)$ defines a probability measure (on which probability space?)

2. *(Durrett 1.1.6)* A set $A \subset \{1, 2, \dots\}$ is said to have **asymptotic density** θ if
   $$\lim_{n\to\infty} \frac{1}{n} |A \cap \{1, 2, \dots, n\}| = \theta\,.$$
   Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists.

   (a) Is $\mathcal{A}$ a σ-algebra? an algebra?

   (b) Is θ additive? σ-additive on $\mathcal{A}$?

3. *Monotone limits*
   Let $\mathcal{A}$ be an algebra. Suppose that $\mathcal{A}$ is closed under countable increasing unions, i.e., $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ whenever $E_j \in \mathcal{A}$ and $E_j \subset E_{j+1}$ for each $j = 1, 2, \dots$.
   Prove that $\mathcal{A}$ is a σ-algebra, i.e., $\mathcal{A}$ is in fact closed under general countable unions.

4. *(Durrett 1.2.3)* Show that a distribution function $F(x) = P(X \leq x)$ has at most countably many discontinuities.

5. (a) *Change of variables (Durrett 1.2.5)*
   Suppose X has continuous density f, that $P(\alpha \leq X \leq \beta) = 1$, and that g is a strictly increasing, differentiable function on (α, β). Show that $Y = g(X)$ has density
   $$\begin{cases} \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}\,, & \text{if } g(\alpha) < y < g(\beta)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$
   In particular, when $g(x) = ax + b$ with $a > 0$, then the density of Y is $\frac{1}{a} f\left(\frac{y-b}{a}\right)$.

   (b) *Log-normal distribution (Durrett 1.2.6)*
   Find the density of $\exp(X)$ when X has a standard normal distribution.

6. (a) *(Durrett 1.2.7)* Suppose X has density function f. Compute the distribution function of $Y = X^2$ and then differentiate it to find its density function.

   (b) *Chi-square distribution*
   Find the density of X^2 when X has a standard normal distribution.