MAT 1600 : Probability I
Assignment 4, due October 12, 2016

19. *(Durrett 2.2.1)* Let $X_1, X_2, \ldots$ be uncorrelated with $E(X_i) = \mu_i$ and $\mathrm{Var}\,(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + \cdots + X_n$ be the n-th partial sum, and $\nu_n = S_n/n$ the corresponding sample average. As $n \to \infty$, show that $\frac{1}{n}S_n - \nu_n \to 0$ in L^2 and in probability.

20. *Monte Carlo integration (Durrett 2.2.3)* Let f be a Borel measurable, Lebesgue integrable function on the unit interval $[0, 1]$. The objective is to construct a probabilistic method for computing the integral

$$I = \int_0^1 f(x)\, dx\,.$$

Let $U_1, U_2, \ldots$ be independent and uniformly distributed on $[0, 1]$, and let

$$I_n = \frac{1}{n}\big(f(U_1) + \cdots + f(U_n)\big)$$

be average of the first n values.

(a) Show that $I_n \to I$ in probability.

(b) If, moreover, $\int |f(x)|^2\, dx < \infty$, use Chebyshev's inequality to estimate

$$P(|I_n - I|) > an^{1/2}\,, \quad \text{for } a > 0\,.$$

Remark. This method for computing integrals can be adapted (by change of variables) to numerically integrate functions of many variables over complicated regions.

21. *(Durrett 2.2.5)* Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i > x) = \frac{e}{x} \log x$ for $x \geq e$.

(a) Show that $E(|X_i| = \infty$.

(b) Construct a sequence of constants $\mu_n \to \infty$ such that $S_n/n = \mu_n \to 0$ as $n \to \infty$.

22. *(Durrett 2.2.6)* (a) Show that if $X \geq 0$ is integer valued then

$$EX = \sum_{n \geq 1} P(X \geq n)\,.$$

(b) Find a similar expression for $E(X^2)$.

23. *Binomial distribution (Durrett 2.1.15)* A random variable X is said to have a Binomial (n, p) distribution if

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k = 0, \dots, n.$$

(a) If X = Binomial (m, p) and Y = Binomial (n, p) are independent, show that $X + Y$ = Binomial $(m + n, p)$.

(b) Look a Example 1.6.3 and use induction to conclude that the sum of n independent Bernoulli (p) random variables is Binomial (n, p).

(c) Use this to compute the mean and variance of the Binomial (n, p) distribution.

24. *Waiting for the next success* Consider a sequence of independent tosses of a biased coin that shows Heads with probability p, and Tails with probability $q = 1-p$. Let X_i be indicator that the ith toss comes up Heads. Let T_n be the number of the toss on which Heads appears for the n-th time.

(a) Find the distribution of T_1, and compute its expectation and variance.

(b) Show that $Y_1 = T_1$ and $Y_2 = T_2 - T_1$ are independent and identically distributed. Use this to compute their covariance.

(c) Express the distribution function of T_2 in terms of suitable Binomial (n, p) random variables. (Do not try to simplify the formula.)

Remark: We will see later that the random variables $Y_n = T_n - T_{n-1}$ are all independent and identically distributed. Their distributions are called (n, p)-negative binomial.