MAT 1600 : Probability I
Midterm Test, October 26, 2016

(Four problems; 20 points each. Time: 90 minutes.)

Please be brief but justify your answers, citing relevant theorems.

1. Let X, Y be independent random variables.

   (a) If the distribution function of X is continuous, does it follow that the distribution function of the sum $X + Y$ is continuous? Why / why not ?

   (b) Suppose X, Y are nonnegative and have probability densities f, g. Find the probability density of the product XY.

2. Consider an infinite sequence of tosses of a single biased coin that produces 'Heads' with probability p. By Kolmogoroff's extension theorem, this defines a probability measure P_p on the product space $\{0, 1\}^{\mathbb{N}}$. Let μ_p be the measure on the unit interval that corresponds to P_p via the binary representation of real numbers.

   (a) If $0 < p < 1$, show that the function $F_p(x) = \mu_p(0, x]$ is continuous.

   (b) Show that the measures μ_p are mutually singular, by finding events A_p for each p such that $\mu_p(A_p) = 1$, but $A_p \cap A_q = \emptyset$ for $p \neq q$. In particular, $\mu_p(A_q) = 0$ when $p \neq q$.

   Hint: Work with the measures P_p and appeal to the Law of Large Numbers (which one?)

3. Let $X_1, X_2, \dots$ be i.i.d. with $P(X_i > x) = e^{-x}$, and let $M_n = \max_{1 \leq m \leq n} X_m$.

   (a) Show that $\limsup_{n \to \infty} \dfrac{X_n}{\log n} = 1$ almost surely.

   (b) Conclude that $\lim_{n \to \infty} \dfrac{M_n}{\log n} = 1$ almost surely.

4. Given a random variable X, define its moment-generating function by

$$\phi(s) = E\left(e^{sX}\right) , \qquad s \in \mathbb{R} .$$

(Note that $\phi(s)$ is always well-defined, but may take the value $+\infty$.)

(a) Show that $\log\phi$ is convex, i.e.,

$$\phi\big((1-\lambda)s + \lambda t\big) \leq \big(\phi(s)\big)^{1-\lambda}\big(\phi(t)\big)^{\lambda}$$

for all $s, t \geq 0$ and all $\lambda \in (0,1)$. (*Hint:* Hölder's inequality.)

(b) Assuming X has finite mean, show that $E(X) \leq \frac{1}{s}\log\phi(s)$ for all $s > 0$.

(c) Prove the Chernoff bound: For any $x > 0$,

$$P(X > x) \leq \inf_{s>0}\left\{e^{-sx}\phi(s)\right\} .$$

(d) Assume that $X_1, X_2, \ldots$ are i.i.d., with moment-generating function $\phi(s)$. Show that their sum $S_n = \sum_{i\leq n} X_i$ satisfies

$$P\left(\frac{S_n}{n} > x\right) \leq \left(\inf_{s>0} e^{-sx}\phi(s)\right)^n .$$