Theorem (Limits of measures are measures). *Let $(P_n)_{n\geq 1}$ be a sequence of measures on $(\Omega, \mathcal{A})$. Suppose that the limit*

$$m(A) := \lim_{n\to\infty} \mu_n(A) < \infty$$

exists for each $A \in \mathcal{A}$. Then m is a measure.
(Of course, if each P_n is a probability measure, then so is m.)

Outline of the proof.

1. *m is finitely additive.*

2. *m is σ-super-additive.*
   Given any sequence of disjoint events $(A_i)_{i\geq 1}$, consider the difference

   $$L := m\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{\infty} m(A_i)\,.$$

   Show that $L \ = \lim m(B_k) \geq 0$, where $B_k = \displaystyle\bigcup_{i=k}^{\infty} A_i$.

   It remains to prove that $L = 0$.

3. *Pass to rapidly converging subsequences.*
   Recursively find subsequences $\tilde{B}_1, \tilde{P}_1, \tilde{B}_2, \tilde{P}_2, \ldots$ such that

   $$|\tilde{P}_n(\tilde{B}_k)| \leq \frac{1}{n} \text{ for } k > n\,, \qquad |\tilde{P}_n(\tilde{B}_k) - m(\tilde{B}_k)| \leq \frac{1}{n\cdot 2^k} \text{ for } n \geq k\,. \tag{1}$$

4. *The key estimate.*
   Set $C = \displaystyle\bigcup_{k=1}^{\infty} \tilde{B}_{2k} \setminus \tilde{B}_{2k+1}$. Then

   $$|\tilde{P}_{2n}(C) - \tilde{P}_{2n-1}(C)| \geq m(\tilde{B}_{2n}) - \frac{4}{n}\,.$$

   Take $n \to \infty$ to see that $L = 0$.