UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL-MAY 2010 EXAMINATIONS
APM351 Y1Y Differential Equations in Mathematical Physics

Examiner: Professor Almut Burchard

Time: 3 hours. No calculators or other aids allowed.
Please try **all six** *problems; total 100 points.*

1. *[20pts]* Consider the **wave equation**

$$\begin{aligned} &u_{tt} = c^2 u_{xx}\,, && 0 < x < 1\,, t > 0\,, \\ &u(0,t) = u(1,t) = 0\,, && t > 0\,, \\ &u(x,0) = 0\,,\ u_t(x,0) = \sin^3(\pi x)\,, && 0 < x < 1\,. \end{aligned} \tag{1}$$

   Hint: Separate variables, and use the triple-angle formula $\sin^3\theta = \frac{1}{4}\big(3\sin\theta - \sin 3\theta\big)$.

   (b) Use the **energy method** to show that the solution is unique.

2. *[10pts]* What is a **characteristic surface** for the wave equation $u_{tt} = c^2 \Delta u$ on $\mathbb{R}^n$? Why are characteristic surfaces important?

3. *[20 pts]* Consider **Burger's equation** $u_t + uu_x = 0$ for $x \in \mathbb{R}$, $t > 0$ with initial condition

$$u(x,0) = \begin{cases} 1 - |x|\,, & -1 \le x \le 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   (a) Write down the characteristic ODE.

   (b) Sketch the characteristics in the x-t-plane for $0 \le t \le 2$. Also sketch the solution $u(\cdot, t)$ at times $t = 0$, 0.5, 1, 1.5, 2.

   When does a shock form? Use the Rankine-Hugoniot condition to derive an equation for the shock. (You need not solve this equation explicitly, but try to make your sketches qualitatively correct.)

   (c) Does your solution satisfy Lax's entropy condition? Please explain!

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4. *[20pts]* We have defined the **Fourier transform** of an integrable function on $\mathbb{R}^n$ by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx\,.$$

(a) Express the Fourier transforms of the following functions in terms of $\hat{f}(k)$:

$$\textit{translation: } \tau_v f(x) = f(x-v)\,, \qquad \textit{scaling: } \sigma_a f(x) = f(x/a)\,.$$

(b) Use the fact that the Gaussian $g(x) = e^{-\pi|x|^2}$ satisfies $\hat{g} = g$ to compute the Fourier transform of $h(x) = |x|^2 e^{-\pi|x|^2}$.

(c) Prove that the Fourier transform of $h(x) = \int_{\mathbb{R}^n} f(y)f(y-x)\, dy$ is nonnegative.

5. *[10pts]* (a) Define the delta **distribution** δ on the real line, and briefly explain its meaning.

(b) Find the first, second, and third distributional derivative of $f(x) = \max\{1-x^2, 0\}$.

6. *[20pts]* For $\varepsilon > 0$, consider the functional

$$\mathcal{I}(u) = \frac{1}{2}\int_0^T \int_0^1 e^{-t/\varepsilon}\left(\varepsilon u_t^2 + u_x^2\right) dxdt\,.$$

(a) Suppose that u minimizes $\mathcal{I}$ among all smooth functions with given boundary conditions. Derive a PDE for u.

Remark: This is the **Euler-Lagrange equation** for $\mathcal{I}$. Note that $\varepsilon = 0$ yields the heat equation.

Let u_ε be the solution of your PDE from (a) with initial and final conditions

$$u(x,0) = 0\,, \quad u(x,T) = \sin(2\pi x)\,,$$

and Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0\,, \quad 0 \le t \le T\,.$$

For $\varepsilon > 0$, it is known that this problem has a unique solution u_ε, which is smooth and satisfies the same maximum principle as Laplace's equation in two dimensions (you are not asked to prove this.)

(b) As $\varepsilon \to 0$, does u_ε converge to a solution of the corresponding problem for the heat equation? Does such a solution even exist? Discuss this question in view of the appropriate maximum principles.

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