UNIVERSITY OF TORONTO
Faculty of Arts and Science

August 2012 EXAMINATIONS
APM351 Y1Y Differential Equations in Mathematical Physics

Examiner: Professor Almut Burchard

Time: 3 hours. No calculators or other aids allowed.
Note the formulas provided at the end of page 2 !
Please try **all seven** *problems; total 100 points.*

1. (10pts) (a) What does it mean for a problem to be **well-posed** ?

   (b) Briefly explain your definition by giving two examples, one well-posed and one ill-posed.

2. (20pts) (a) What are the defining properties of the **Green's function** for a domain $D \subset \mathbb{R}^3$?

   (b) How can the Green's function be used to solve the **Poisson problem**

   $$\Delta u = 0 \quad \text{on } D\,, \qquad u|_{\partial D} = g\,?$$

   (c) Construct the Green's function for the upper unit half-ball in $\mathbb{R}^3$. Please prove that your construction yields the defining properties !
   Hint: The Green's function for the unit ball is $G_1(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x|\,|\bar{x}-y|}$, where $\bar{x} = \frac{x}{|x|^2}$.

3. (10pts) State the **mean value property** for harmonic functions, and show that it implies the strong maximum principle.

4. (10pts) What is a **characteristic surface** for the wave equation $u_{tt} = c^2 \Delta u$ on $\mathbb{R}^n$? Why are characteristic surfaces important?

5. (20pts) Suppose that $u(x,t)$ solves the heat equation

   $$u_t = u_{xx}\,, \quad u(x,0) = \phi(x)\,,$$

   where the initial condition ϕ is a smooth test function, and let $\hat{u}(k,t)$ be its **Fourier transform**.

   (a) Derive an equation for $\frac{d}{dt}\hat{u}(k,t)$.

   (b) Solve the differential equation from (a) to obtain a formula for $\hat{u}(k,t)$ in terms of $\hat{\phi}(k)$.

   (c) Transform back and explicitly evaluate the resulting dk-integral to obtain an integral formula for $u(x,t)$ in terms of ϕ.
   Hint: What is the inverse Fourier transform of $g(k) = e^{-4\pi^2 k^2}$?

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6. (15pts) (a) Compute the first four distributional derivatives of $f(x) = |\sin x|$ on the real line.

   (b) Let (f_n) be a sequence of distributions on $\mathbb{R}$ that converges weakly to a distribution f. Does it follow that the distributional derivatives f_n' converge weakly to f'? Please prove your claim !

7. (15pts) Consider **Burger's equation** $u_t + uu_x = 0$ for $t > 0$ with initial condition

$$u(x,0) = \begin{cases} 0 & \text{for } 0 < x < 1\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

   (a) Write down the characteristic equations.

   (b) State the Rankine-Hugoniot jump condition and the Lax entropy condition. How many shocks are there? Determine their location, height, and speed up to time $t = 2$. How do these change over time after $t = 2$? What will happen as $t \to \infty$?

   (c) Sketch the characteristics and the solution below.
   Note: Do not solve the equation explicitly, but try to make your pictures qualitatively correct.

Useful formulas

- The fundamental solution of the **diffusion equation** on $\mathbb{R}^n$ is $\Phi(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.
- The fundamental solution of **Laplace's equation** on $\mathbb{R}^3$ is $G_0(x,y) = -\frac{1}{4\pi|y-x|}$.
- We have defined the Fourier transform on $\mathbb{R}^n$ by $\hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y} f(y)\, dy$.
  With this convention, the Gaussian $f(x) = e^{-\pi|x|^2}$ satisfies $\hat{f} = f$.

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