APM 351: Differential Equations in Mathematical Physics
Test 1, November 22 2010

(Six problems, total 110pts.)

1. (20pts) (a) Use the *Method of Characteristics* to solve the partial differential equation

$$u_x + u_y + u = 0 \tag{1}$$

   with initial values $u(x,0) = x^2$. Is the solution defined on all of $\mathbb{R}^2$? Is it unique?

   (b) Consider Eq. (1) with initial conditions given by a smooth curve $\Gamma(s) = \begin{pmatrix} x_0(s) \\ y_0(s) \\ u_0(s) \end{pmatrix}$.

   What does the Existence and Uniqueness Theorem say about this problem? Please remember to state the assumptions as well as the conclusions!

2. (10pts) In which regions of the plane is the Tricomi equation $yu_{xx} - u_{yy} = 0$
   (a) *elliptic* ?   (b) *hyperbolic* ?

3. (20pts) Use *Separation of Variables* to find special bounded solutions of the heat equation

$$u_t = u_{xx}$$

   for $x \in \mathbb{R}$, $t > 0$. *(Please discard unbounded solutions.)*

4. (20pts) Consider the wave equation

$$u_{tt} + au_t = u_{xx}\,, \quad 0 < x < 1, t > 0$$

   with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ and smooth initial values

$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,.$$

   (a) If $a > 0$, use the *energy* $E(t) = \frac{1}{2}\int_0^1 u_t^2 + u_x^2\,dx$ to show that there can be at most one solution.

   (b) What can you say when $a < 0$?

5. (20pts) Let $u(x) = 1$ for $0 \le x \le \frac{\pi}{2}$, and $u(x) = 0$ for $\frac{\pi}{2} < x < \pi$.

   (a) In the *Fourier sine series*

$$u(x) = \sum_{k=1}^{\infty} A_k \sin kx\,,$$

   determine the coefficients A_k.

   (b) Briefly comment in which sense the series converges.