APM 351: Differential Equations in Mathematical Physics
Test 1, November 4 2011

(Five problems; 80 points total)

1. (20 pts) Consider the partial differential equation $u_{tt} - 3u_{xt} + 2u_{xx} = 0$.

   (a) Is the equation *elliptic*, *parabolic*, or *hyperbolic*?

   (b) Find the general solution. (*Hint:* You can factor the differential operator.)

   (c) Given initial values $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, does there *exist* a *unique solution* with these initial values? Please justify your answer briefly! (You need not find the formula).

2. (10 pts) Suppose that u solves the *wave equation* $u_{tt} = u_{xx}$ with initial values $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$. If ϕ vanishes outside $[-3,-1]$ and ψ vanishes outside $[1,3]$, where does the solution have to vanish? (A sketch would be helpful.)

3. (20pts) (a) Write down a solution of the *diffusion equation*

   $$u_t = u_{xx}\,, \quad (x \in \mathbb{R}, t > 0)$$

   with initial values $u(x,0) = \max\{x, 0\}$. (Do not evaluate the integral).

   (b) Show that your solution is strictly positive for all $x \in \mathbb{R}$ and all $t > 0$.

   (c) Show that for every $t > 0$, the solution is strictly increasing in x.

   (d) State the *maximum principle*. What does it imply for the solution you found in part (a)? How does it relate to your results from parts (b) and (c)?

4. (20pts) (a) Use *separation of variables* to find special solutions of the diffusion equation

   $$\begin{array}{ll} u_t = u_{xx}\,, & 0 < x < \frac{\pi}{2},\ t > 0, \\ u(0,t) = u_x(\frac{\pi}{2}, t) = 0\,, & t > 0\,. \end{array}$$

   (Please note the mixed boundary conditions).

   (b) Are the eigenfunctions X_n you found in part (a) orthogonal? Are the eigenvalues real? Are they positive? What is the underlying reason?

5. (10pts) Expand the function $f(x) = 1$ for $x \in (0, \pi)$

   (a) as Fourier cosine series;

   (b) as a Fourier sine series.