APM 351: Differential Equations in Mathematical Physics
Test 2, March 2, 2010

(Choose five out of six problems; 20 points each. Time: 3 hours.)

1. (a) What are the defining properties of a **Green's function** for a domain $U \subset \mathbb{R}^2$?

   (b) Let $G(x, y)$ be the Green's function for U. How can G be used to solve Poisson's equation

$$\begin{cases} -\Delta u = f\,, & \text{in } U\,, \\ u = 0\,, & \text{on } \partial U\ ? \end{cases}$$

   (c) Construct the Green's function for the Laplacian on the positive quadrant

$$\{(x_1, x_2) \in \mathbb{R}^2 \ \mid x_1, x_2 > 0\}\,.$$

   Please justify why your construction yields the required properties!

2. Write down an explicit formula for the solution of the modified **diffusion equation**

$$\begin{cases} u_t - \Delta u + cu = f & x \in \mathbb{R}^n\,, t > 0 \\ u(x, 0) = \phi(x) & x \in \mathbb{R}^n, t > 0 \end{cases}$$

   Hint: Rewrite the equation for $v(x, t) = e^{ct}u(x, t)$, and use Duhamel's formula.

3. Consider a solution u of the **wave equation**

$$\begin{cases} u_{tt} = \Delta u\,, & x \in \mathbb{R}^3, t > 0 \\ u(x, 0) = 0, u_t(x, y) = \psi(x)\,, & x \in \mathbb{R}^3\,. \end{cases}$$

   Assume that ψ is smooth, bounded, and vanishes for $|x| > 1$.

   (a) Where does u have to vanish? (A sketch would be useful.)

   (b) Argue that $u(x, t) = O(t^{-1})$ *uniformly* as $t \to \infty$; that is, show that

$$\sup_{x,t}\{t \cdot |u(x, t)|\} < \infty\,.$$

   Hint: A spherical cap of radius r and opening angle ϕ has area $2\pi r^2(1-\cos\phi) \leq Const.\, r^2\phi^2$.

4. (a) Construct a basis for the **spherical harmonics** (harmonic polynomials) of degrees 0, 1, 2, and 3 in three variables (x, y, x). Briefly explain your method.

   (b) Use Part (a) to solve Laplace's equation on the unit ball $B = \{(x, y, z) : x^2+y^2+z^2 < 1\}$,

$$\begin{cases} \Delta u = 0\,, & \text{in } B\,, \\ u(x, y, z) = x^2\,, & \text{on } \partial B\,. \end{cases}$$

5. Consider the eigenvalue problem for the Neumann Laplacian on the two-dimensional unit disc,

$$\begin{cases} -\Delta u = \lambda u\,, & |x| < 1 \\ \frac{\partial u}{\partial n} = 0\,, & |x| = 1\,. \end{cases}$$

(a) Use **Separation of Variables** to split the problem into two eigenvalue problems.

(b) Solve the angular problem.

(c) Consider, as a special case, solutions of the form $u(r)$ (i.e., purely radial solutions where the angular part is constant.) Express these solutions in terms of the zeroth order Bessel function $J_0(r)$. Please explain your reasoning!

6. Let $U \subset \mathbb{R}^3$ be a bounded, connected open set with smooth boundary.

(a) Write down the **Rayleigh principle** for the lowest eigenvalue of the Laplacian with Dirichlet boundary conditions.

(b) Write down a corresponding (minimax or maximin) principle for the n-th eigenvalue λ_n.

(c) Show that the eigenvalues of the Neumann Laplacian lie below the corresponding eigenvalues for the Dirichlet Laplacian.

(d) What does Weyl's law say about the behavior of the eigenvalues λ_n as $n \to \infty$?

Useful formulas.

- The fundamental solution of **Laplace's equation** on $\mathbb{R}^2$ is $G_0(x) = -\frac{1}{2\pi}\log|x|$.
- The source function of the **diffusion equation** $u_t = \Delta u$ in $\mathbb{R}^n$ is $S(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.
- **Duhamel's formula:** The solution of the linear equation $\dot{y} = Ay$ is given by

$$y(t) = e^{At}y(0) + \int_0^t e^{A(t-s)} f(s)\, ds\,.$$

- The **Bessel functions** $J_n(r)$ are the unique bounded solutions of Bessel's equation

$$J'' + \frac{1}{r}J' + \left(1 - \frac{n^2}{r^2}\right) J = 0$$

for $n = 1, 2, \dots$. Each J_n is a smooth function that changes sign at an infinite sequence of zeroes $z_{n,1}, z_{n,2}, \dots \to \infty$, separated by an infinite sequence of critical points $p_{n,1}, p_{n,2}, \dots$.

- **Kirchhoff's formula:** The solution of the three-dimensional wave equation $u_{tt} = c^2\Delta u$ with initial values (ϕ, ψ) is given by

$$u(x_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi t_0}\int_{|x-x_0|=t_0} \phi(x)\, dS(x)\right] + \frac{1}{4\pi t_0}\int_{|x-x_0|=t_0} \psi(x)\, dS(x)\,.$$