APM 351: Differential Equations in Mathematical Physics
Test 2, January 20, 2012

(Five problems; 80 points total)

1. (20pts) (a) Find the full Fourier series $\sum_{k=-\infty}^{\infty} a_k e^{ikx}$ of the function
$$f(x) = e^x\,, \qquad x \in (0, 2\pi)\,.$$
Does the series converge in L^2? pointwise? uniformly? Please explain your reasoning!

   (b) What does Parseval's identity give for the Fourier coefficients of f?

   (c) What it the k-th Fourier coefficient of f'? Why is it different from ika_k?

2. (10pts) Solve $\Delta u = 0$ on a three-dimensional spherical shell $r_0 < |x| < R$ with boundary conditions $u = A$ at $|x| = r_0$ and $u = B$ at $|x| = R$, where A and B are given constants.

   Hint: Look for a radial solution $u(x) = v(|x|)$.

3. (20pts) Assume that u_0 solves Poisson's equation
$$\begin{aligned} \Delta u_0 &= f\,, && \text{on } D\,,\\ u_0(x) &= g(x)\,, && \text{for } x \in \partial D \end{aligned}$$
on a smooth bounded domain $D \subset \mathbb{R}^n$. Prove that u_0 minimizes
$$\mathcal{I}(u) := \frac{1}{2}\int_D |\nabla u|^2\,dx + \int_D f(x)u(x)\,dx$$
among all smooth functions u on D that satisfy the boundary conditions $u(x) = g(x)$ on ∂D.

   Hint: Write $u = u_0 + v$, and try to show that $\mathcal{I}(u_0 + v) - \mathcal{I}(u_0) \geq 0$. Expand the square and use Green's first identity ...

4. (20pts) Consider the diffusion equation
$$u_t = u_{xx} + f(x,t)\,, \qquad 0 < x < 1, t > 0\,,$$
with initial and boundary conditions
$$u(0,t) = u(1,t) = 0 \text{ for } t > 0, \qquad u(x,0) = g(x) \text{ for } 0 < x < 1\,.$$
(a) Use the maximum principle to show that the solution of this problem is unique.

   (b) Give a second proof of uniqueness, using the energy method.

5. (10pts) Let f be a smooth 2π-periodic function. Show that
$$||f'||_{L^2}^2 \leq ||f||_{L^2} \cdot ||f''||_{L^2}\,.$$