MAT 351: Partial Differential Equations
Assignment 11, due January 16, 2017

Summary

The **fundamental solution** of the Laplacian in $\mathbb{R}^n$ is given by

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x|\,, & n=2\,,\\ -\frac{1}{n(n-2)\omega_n|x|^{n-2}}\,, & n\geq 3\,,\end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. In three dimensions $\Phi(x) = -\frac{1}{4\pi|x|}$ can be interpreted as the gravitational potential of a point mass, or equivalently, the electrostatic field of a point charge at the origin.

If f is a bounded function on $\mathbb{R}^n$ (where $n \geq 3$) that vanishes outside some ball, then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy\,.$$

is the unique solution of Poisson's equation

$$\Delta u = f\,, \quad x \in \mathbb{R}^n$$

with $u(x) \to 0$ as $|x| \to \infty$. (Normalizing the potential to vanish at infinity is a standard choice in Physics. There are many other solutions of Poisson's equation, all of which which grow at infinity.) We say that

$$\Delta\Phi = -\delta$$

in the sense of distributions.

A similar formula holds for Poisson's eqation on a bounded domain in $\mathbb{R}^n$: The unique solution of the Dirichlet problem

$$\Delta u = f\,, \quad \text{for } x\in D\,, \qquad u(x) = g(x)\,, \quad \text{for } x \in \partial D$$

is given by

$$u(y) = \int_D G(y,x)f(y)\,dy + \int_{\partial D} g(y)\nabla_y G(y,x)\cdot n(y)\,dS(y)\,.$$

Here, $G(y,x)$ is the **Green's function** of the domain. It is defined by the properties that

- $G(y,x) - \Phi(x,y)$ is smooth and harmonic on D;
- $G(y,x) = 0$ for $y \in \partial D$

for every $x \in D$. We will see that the Green's function is negative and symmmetric, i.e.,

- $G(x,y) = G(y,x)$.

The function defined on the boundary of D by

$$P(x,y) = \nabla_y G(x,y) \cdot n(y)$$

is called the **Poisson kernel** associated with D.

The proofs in this section are based on **Green's identities:** For any pair of smooth functions u, v on D, we have

$$\int_D v\Delta u\,dx = -\int_D \nabla u \cdot \nabla v\,dx + \int_{\partial D} v\nabla u \cdot n(x)\,dS(x)\,, \tag{1}$$

$$\int_D (u\Delta v - v\Delta u)\,dx = \int_{\partial D} (u\nabla v - v\nabla u) \cdot n(x)\,dS(x)\,. \tag{2}$$

Assignments:

Read Chapter 7 of Strauss.

Hand-in (due Monday, January 16):

45. Find the radial solutions (depending only on $r = |x|$) of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant.
(*Hint:* Substitute $u(r) = \frac{v(r)}{r}$. Solutions may blow up at $r = 0$.)

46. Use the Mean Value Property of harmonic functions in n variables to derive the maximum principle. Conclude that the solution of Poisson's problem $\Delta u = f$ on a bounded domain D, with Dirichlet boundary conditions $u|_{\partial D} = g$ is unique (assuming it exists).

47. Let D be an open set with smooth boundary in $\mathbb{R}^n$. Use the divergence theorem to show that the Neumann problem

$$\Delta u = f \text{ in } D\,, \quad \nabla u \cdot \nu = g \text{ on } \partial D$$

cannot have a solution unless $int_D f\,dx = \int_{\partial D} g\,dS$.

48. **Dirichlet's principle for Neumann boundary conditions** *(Strauss, Problem 7.1.5)*
Prove that among *all* real-valued functions w on D, the quantity

$$E(w) = \frac{1}{2}\int_D |\nabla w|^2\,dx - \int_{\partial D} hw\,ds$$

is minimized by $w = u$, where u is a harmonic function that satisfies the Neumann boundary condition $\nabla u \cdot n|_{\partial D} = h$. Here, h is a given function on ∂D with $\int_{\partial D} h\,dS = 0$.

49. Consider a homogeneous polynomial in two variables

$$P(x,y) = a_0 x^k + a_1 x^{k-1} y + \cdots + a_k y^k\,.$$

(a) Under what conditions on the coefficients is the polynomial harmonic? How many linearly independent harmonic polynomials are there of degree k?

(b) Write down a basis of the space of harmonic polynomials of degree $k \leq 4$, in both Cartesian and polar coordinates. Identify them as the real (or imaginary) parts of holomorphic functions.