MAT 351: Partial Differential Equations
Assignment 13, due February 6, 2017

Summary

Consider the wave equation $u_{tt} = c^2 \Delta u$, with initial conditions $u(x,0) = \phi(x)$ and $u_y(x,u) = \psi(x)$. Two important properties of the wave equation in any dimension are:

- The **energy** $E(t) = \frac{1}{2} \int u_t(x,t)^2 + c^2 |\nabla u(x,t)|^2 \, dx$ is **conserved** (constant in time);
- **Causality:** The solution $u(x,t)$ depends on the initial conditions only on the ball $B_{c|t|}(x)$. In other words, the domain of dependence of (x,t) is the solid backwards **light cone**

$$\left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} \ \mid \ s \le t, |y-x|^2 \le c^2 |t-s|^2 \right\}.$$

We will restrict attention to the cases of two and three spatial dimensions appearing in classical Physics. In three dimensions, the solution of the wave equation is given by **Kirchhoff's formula**

$$u(x,t) = \frac{\partial}{\partial t} \left\{ \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \phi(y) \, dS(y) \right\} + \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \psi(y) \, dS(y) \, .$$

Remarkably, the solution depends on the initial data only on the (surface of the) light cone, i.e., waves travel exactly at the speed of light. This is called **Huygens principle**. It is typical for solutions of the wave equation in all odd dimensions $n = 2k+1 \ge 3$.

In two dimensions, we have **Poisson's formula**

$$u(x,t) = \frac{\partial}{\partial t} \left\{ \frac{1}{2\pi c} \int_{|y-x|<ct} \frac{\phi(y)}{(c^2t^2 - |y-x|^2)^{1/2}} \, dy \right\} + \frac{1}{2\pi c} \int_{|y-x|<ct} \frac{\psi(y)}{(c^2t^2 - |y-x|^2)^{1/2}} \, dy \, .$$

Note that Huygens' principle fails in two dimensions (and generally in even dimensions.)

The key to the proof of Kirchhoff's formula is the observation that the **spherical mean** of a solution, given by

$$\bar{u}(r,t;x) = \frac{1}{n\omega_n r^{n-1}} \int_{|y-x|=r} u(y,t) \, ds(y)$$

satisfies **Darboux' equation**

$$u_{tt} = c^2 \left(u_{rr} + \tfrac{n-1}{r} u_r \right) \, .$$

(Here, the denominator $n\omega_n$ is the $n-1$-dimensional surface area of the n-ball. In $n=3$ dimensions, its value is 4π.) Darboux's equation can be solved explicitly when n is odd, and Kirchhof's formula follows by setting $u(x,t) = \bar{u}(0,t;x)$. From there, we obtain the solution in even dimensions by using **Hadamard's method of descent**.

Assignments:

Read Sections 9.1-3 of Strauss.

53. (a) A **plane wave** is a solution of the wave equation of the form $u(x,t) = f(k \cdot x - ct)$, where f is a C^2-function. Find all the three-dimensional plane waves.

    (b) Verify that $u(x,t) = (c^2t^2 - |x|^2)^{-1}$ satisfies the three-dimensional wave equation except on the light cone.

54. (a) Use Kirchhoff's formula to find the solution of the three-dimensional wave equation with initial data $u(x,u) = 0$, $u_t(x,0) = x_2$. (*Hint:* $\Psi(x) = x_2$ has the mean value property.)

    (b) Use the Darboux equation (for radial solutions of the wave equation) to solve the three-dimensional wave equation with initial data $u(x,0) = 0$, $u_t(x,u) = |x|^2$.

55. Consider the **Klein-Gordon equation** $u_{tt} - c^2\Delta u + m^2 u$, where $m > 0$.

    (a) What is the energy? Show that it is conserved.

    (b) Prove the causality principle for it.

56. Consider the one-dimensional wave equation $u_{xx} = c^2 u_{xx}$ with initial values given on a surface $\mathcal{S} = \{(x,t) \mid t = \gamma(x)\}$, by

$$u((x,\gamma(x)) = \phi(x)\,, \quad \frac{\partial u}{\partial n} = \Psi(x)\,.$$

    If $\mathcal{S}$ is space-like, i.e., $|\gamma'(x)| < \frac{1}{c}$, prove that the initial-value problem has a unique solution. (*Hint:* The solution can be written as $u(x,t) = F(x+ct) + G(x-ct)$.)

57. Thinking of space-time as $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, let Γ be the diagonal 4×4 matrix with diagonal entries $1, 1, 1, -1$. A **Lorentz transformation** is an invertible matrix that satisfies $L^t \Gamma L = \Gamma$, or equivalently, $L^{-1} = \Gamma L^t \Gamma$.

    (a) Prove that Lorentz transformations form a group, i.e., products and inverse of Lorentz transformations are again Lorentz transformations. What can you say about the determinant of L?

    (b) Show that L is Lorentz if and only if it preserves the quadratic form $m(x,t) = |x|^2 - t^2$, i.e., $m(L(v)) = m(v)$ for all $v = (x,t) \in \mathbb{R}^4$. The quadratic form m is called the **Lorentz metric**.

    (c) If L is a Lorentz transformation, and $U(z) = u(L(z))$, show that

$$u_{tt} - \Delta u = U_{tt} - \Delta U\,,$$

    i.e., if u solves the wave equation, so does U.

    (d) Explain the meaning of a Lorentz transformation in more geometrical terms, by drawing an analogy to the group of orthogonal matrices. How does m relate to the light cone?