MAT 351: Partial Differential Equations
Assignment 14, due February 13, 2017

Please note: This week's tutorial (February 10) is canceled.

Summary:

For the wave equation $u_{tt} = c^2\Delta u$, the heat equation $u_t = k\Delta u$, and the Schrödinger equation $iu_t = -\Delta u$, separation of variables leads to the same eigenvalue problem

$$-\Delta u = \lambda u\,.$$

It turns out that this eigenvalue problem has no solutions on $\mathbb{R}^n$ that decay at infinity or are even square integrable. (For every vector k, the function $u(x) = e^{-ik\cdot x}$ is a bounded solution with $\lambda = |k|^2$ but these don't lie in L^2.) So we had to investigate other methods of solutions.

- The solutions of the **wave equation** in one, two, and three spatial dimensions are given by the formulas of D'Alembert, Poisson and Kirchhoff. Similar formulas can be derived in higher dimensions.
- The solution of the **heat equation** with $u(x,u) = \phi(x)$ is given by

  $$u(x,t) = (4\pi kt)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4kt}}\phi(y)\,dy\,.$$

  The positivity of the heat kernel $(4\pi kt)^{-n/2}e^{-\frac{|x|^2}{4kt}}$ is a manifestation of the maximum principle.

  This formula remains valid, if k is a complex number with positive real part, provided that we take the square root $\sqrt{k}$ to have positive real part. The integral converges and defines a smooth function, so long as ϕ is bounded and integrable.
- By analytic continuation to $k = i$, we obtain for the **Schrödinger equation** the solution formula

  $$u(x,t) = (4\pi it)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}}\phi(y)\,dy\,.$$

  Here, the square root in the first factor should be chosen as $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. Note that the integral is now oscillatory, and will diverge unless ϕ itself decays at infinity. This is related to the wave-like properties of Schrödinger's equation.

  The fact that the kernel $(2\pi it)^{-n/2}e^{-\frac{|x-y|^2}{4it}}$ never vanishes indicates infinite speed of propagation.

The above formulas are valid on the entire space, $\mathbb{R}^n$. We now return to the study of these equations on finite domains, with suitable boundary conditions, and revisit the Separation of Variables technique. In many applications, it is useful to work in polar coordinates and separate the radial from the angular variables. The resulting equations for the radial dependence give rise to special functions, such as Bessel functions. The angular part will be solved by spherical harmonics.

Assignments:

Complete Chapter 9 of Strauss and move on into Chapter 10.

58. (a) Derive the conservation of energy for the wave equation on a domain D with Neumann boundary conditions.

    (b) Solve the wave equation in the square $(0, \pi) \times (0, \pi)$ with homogeneous Neumann conditions on the boundary, and initial conditions $u(x, y, 0) = \sin^2 x$, $u_t(x, y, 0) = 0$.

    (c) Verify that conservation of energy is indeed valid for your solution.

59. For the wave equation in two space dimensions, find all solutions of the form

$$u(x, t) = e^{-i\omega t} f(|x|, t)\,.$$

60. Consider the heat equation

$$u_t = k\Delta u\,, \quad u(x, 0) = \phi(x)$$

for $x \in \mathbb{R}^n$ and $t > 0$. Here, ϕ is a continuous function with compact support on $\mathbb{R}^n$.

For $n = 1$, we have shown that

$$u(x, t) = \int_{\mathbb{R}} S(x - y)\phi(y)\, dy\,,$$

where $S(x, t) = (4\pi kt)^{-1/2} e^{-x^2/(4kt)}$ is the fundamental solution.

(a) Use the one-dimensional case to solve the heat equation in the case where the initial values are given by a product of continuous functions with compact support

$$\phi(x) = \prod_{i=1}^{n} \phi_i(x_i)\,.$$

(b) Use Weiserstrass' approximation theorem to argue that the formula you found holds more generally for every continuous function ϕ with compact support.