MAT 351: Partial Differential Equations
Assignment 16, due March 6, 2017

A **spherical harmonic** of degree ℓ is a function Y on the unit sphere in $\mathbb{R}^n$, such that

$$P(x_1, \ldots, x_n) = r^\ell Y(\omega)$$

is a harmonic homogeneous polynomial of degree ℓ. Here $r = |x|$ is the radius, and $\omega = \frac{x}{|x|}$ is the direction vector. Spherical harmonics are useful for solving radially symmetric problems in $\mathbb{R}^n$, such as finding the eigenvalues of the Laplacan on a ball, or the eigenstates of the hydrogen atom in quantum mechanics.

We first construct a basis for the space of harmonic homogeneous polynomials $P(x_1, \ldots, x_n)$ of degree ℓ. To do this, we expand such a polynomial in terms of the n-th variable

$$P(x_1, \ldots, x_n) = \sum_{k=0}^{\ell} p_k(x_1, \ldots, x_{n-1}) x_n^k ,$$

where each p_k is a homogeneous polynomial of degree $\ell - n$. Setting

$$\Delta P = 0$$

yields the recursion

$$k(k-1)p_k(x_1, \ldots, x_{n-1}) + \Delta p_{k-2}(x_1, \ldots, x_{n-1}) = 0 , \quad k = 2, \ldots, \ell.$$

Thus, $P(x_1, \ldots, x_n)$ is determined uniquely by specifying two polynomials $p_0(x_1, \ldots, x_{n-1})$ of degree ℓ, and $p_1(x_1, \ldots, x_{n-1})$ of degree $\ell - 1$.

- $n = 2$ **variables:** We write

  $$P(x, y) = \sum_{k=0}^{\ell} p_k(x) y^k ,$$

  where $p_k(x) = a_k x^{\ell-k}$. For example, when $\ell = 3$,

  $$\begin{array}{lcl} p_0 = x^3, p_1 = 0 & \text{gives} & P = x^3 - 3xy^2 , \\ p_0 = 0, p_1 = x^2 & \text{gives} & P = x^2 y - xy^2 . \end{array}$$

  In general, choosing (for $\ell \geq 1$)

  $$\begin{array}{rcl} p_0 = x^\ell, p_1 = 0 & \text{gives} & P = x^\ell - \frac{\ell(\ell-1)}{2} x^{\ell-2} y^2 + \ldots , \\ p_0 = 0, p_1 = x^{\ell-1} & \text{gives} & P = x^{\ell-1} y - \frac{(\ell-1)(\ell-2)}{6} x^{\ell-3} y^3 + \cdots , \end{array}$$

  and we obtain a basis for the space of homogeneous harmonic polynomials of degree ℓ. For $\ell \geq 1$, this space has dimension 2.

  Alternately, we can use the basis $\{\mathbf{Re}(x + iy)^k, \mathbf{Im}(x + iy)^k\}$. This yields for the space of spherical harmonics of degree ℓ in $\mathbb{R}^2$ the basis $\{\cos(\ell\theta), \sin(\ell\theta)\}$.

- $n = 3$ **variables:** Here,

$$P(x,y,z) = \sum_{k=0}^{\ell} p_k(x,y) z^k\,,$$

where $p_k(x,y)$ has degree $\ell - k$. Again, we get to choose p_0 and p_1, and use the recursion to determine $p_k, \dots, p_\ell$. For example, when $\ell = 3$,

$$\begin{array}{rcl} p_0 = x^3, p_1 = 0 & \text{gives} & P = x^3 - 3xz^2\,, \\ p_0 = x^2y, p_1 = 0 & \text{gives} & P = x^2y - yx^2\,, \\ p_0 = 0, p_1 = x^2 & \text{gives} & P = x^2z - xz^2\,. \end{array}$$

The harmonic polynomials of degree ℓ form a vector space of dimension $2\ell + 1$. The spherical harmonics are given by the functions $Y(\theta,\phi) = P(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Notice that the basis we constructed here is different from the basis $\{Y_{\ell,m}(\theta,\phi), m = -\ell, \dots, \ell\}$ that appears in many Physics textbooks.

Theorem. *(1.) Spherical harmonics of different degree are orthogonal in $L^2(S^{n-1})$.*
(2.) $L^2(S^{n-1})$ has an orthonormal basis consisting of spherical harmonics.
(3.) Every polynomial in n variables is a linear combination of polynomials $|x|^{2k}P(x)$, where k is an integer and P is a harmonic polynomial.

Assignments:

Remind yourself of harmonic polynomials and spherical harmonics. Section 10.7 of Strauss offers an alternate approach to spherical harmonics on three diemenions, in terms of spherical coordinates.

1. (a) Consider Schrödinger's equation with a radial potential $V(|x|)$ on $\mathbb{R}^2$ in polar coordinates

$$iu_t = -\frac{1}{2}\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) + V(r)u\,.$$

Separate variables to obtain special solutions of the form $u = T(t)R(r)\Theta(\theta)$.

(b) Assuming that $V(r) = r^2$, substitute $R(r) = e^{-\frac{r^2}{2}} r^{|m|} P(r)$. Write down a differential equation for P, and explain how to obtain polynomial solutions.
Remark: Only these polynomial solutions yield eigenfunctions that lie in L^2 (why?)

2. Consider the eigenvalue problem for the Neumann Laplacian on the two-dimensional unit disc,

$$\begin{cases} -\Delta u = \lambda u\,, & |x| < 1 \\ \frac{\partial u}{\partial n} = 0\,, & |x| = 1\,. \end{cases}$$

As in the previous problem, this can be split into two eigenvalue problems, and the angular problem is solved explicitly by the functions e^{inx}, where n is an integer.

(a) Write down the radial problem corresponding to a given value of λ and n. Remember to state the boundary conditions !

(b) Given n, express the solutions of the radial in terms of the n-th order Bessel function $J_n(r)$. Please explain your reasoning (a picture may help.)