MAT 351: Partial Differential Equations
Assignment 18, due March 27, 2017

Summary:

- A **test function** on a domain $D \subset \mathbb{R}^d$ is a smooth function with compact support in D.

The space of test functions on $\mathbb{R}^d$ will be denoted by $\mathcal{D}$. We say that $\lim \phi_j = \phi$ in $\mathcal{D}$, if *(i)* the functions are supported on a common compact set $K \subset \mathbb{R}^d$, *(ii)* the functions converge uniformly to ϕ, and *(iii)* all their derivatives converge uniformly as well.

- A **distribution** is a linear transformation mapping test functions to $\mathbb{R}$ (or $\mathbb{C}$).

In other words, a distribution f assigns to each $\phi \in \mathcal{D}$ a scalar $f(\phi)$. We require that this transformation be **continuous** on $\mathcal{D}$, in the sense that

$$\lim \phi_j = \phi \quad \Rightarrow \quad \lim f(\phi_j) = f(\phi)\,.$$

We denote the space of distributions by $\mathcal{D}'$, and think of it as the dual space of $\mathcal{D}$.

- A sequence of distributions $\{f_j\}$ **converges weakly** to f, if $\lim f_j(\phi) = f(\phi)$ for all $\phi \in \mathcal{D}$.

Functions are important special cases of distributions. If f is a continuous function on $\mathbb{R}^d$, we can define the corresponding distribution by

$$f(\phi) = \int f(x)\phi(x)\,dx\,.$$

We often write distributions in this form, even when they are not given by a function. For example, the Dirac δ-distribution is defined by

$$\delta(\phi) = \int \phi(x)\delta(x)\,dx := \phi(0)\,.$$

The δ-distribution on $\mathbb{R}^d$ can be obtained as the weak limit of a **Dirac sequence** $\varepsilon^{-d} f(\varepsilon^{-1}x)$, where f is a nonnegative integrable function with $\int f(x)\,dx = 1$, and $\varepsilon \to 0^+$.

- **Distributional derivatives** of f are defined by $\big(D_i f\big)(\phi) = -f\big(\frac{\partial}{\partial x_i}\phi\big)$ for all test functions ϕ.

Distributional derivatives are also called **weak derivatives**. If f is given by a differentiable function, then its distributional derivatives are given by the classical derivatives of f. To give another example, the derivative of the δ-distribution is one dimension is defined by $\delta'(\phi) = -\phi'(0)$.

When solving a linear PDE $Lu = 0$, it is often useful to consider distributional solutions. For example, the fundamental solution of Laplace's equation $\Delta u = f$ on $\mathbb{R}^d$, given by

$$G_0(x) = C_d |x|^{2-d}\,,$$

(where C_d is a specific dimension-dependent constant) solves

$$-\Delta u = \delta$$

in the sense of distributions. In this case, both G_0 and its gradient ∇G_0 turn out to be functions. But note that distributional solutions make no sense for nonlinear equations, because a nonlinear function of a distribution is not a distribution. (For example, δ^2 has no meaning.)

Assignments:

Read the first two sections of Chapter 12 of Strauss.

73. Compute the first three distributional derivatives of the function

$$f(x) = \max\{0, 1 - x^2\}\,.$$

74. Let f be a distribution on $\mathbb{R}$ with $f' = 0$.

    (a) What does that mean?

    (b) Prove that $f(\phi) = 0$ for all test functions ϕ with $\inf_{\mathbb{R}} \phi(x)\,dx = 0$.

    (c) Conclude that f is given by a constant *function* $f(x) = c$, by showing that

    $$f(\phi) = c \int_{\mathbb{R}} \phi(x)\,dx$$

    for all test function ϕ. (*Hint:* First consider the case where $\int_{\mathbb{R}} \phi(x)\,dx = 0$.)

75. Consider **Burger's equation**

$$u_t + uu_x = 0\,, \quad u(x,0) = u_0(x) \tag{1}$$

for $x \in \mathbb{R}$ and $t > 0$. A **integral solution** of the equation is a function u such that

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + \frac{1}{2}u^2\phi_x\,dxdt + \int_{-\infty}^\infty u_0(x)\phi(0,x)\,dx \tag{2}$$

holds for every smooth test function $\phi(x,t)$ with compact support in $\mathbb{R} \times [0, \infty)$.
(Note that ϕ need not vanish on the line $t = 0$.)

    (a) Suppose u itself is smooth. Verify that then Eq. (2) and Eq. (1) are equivalent.

    (b) Let u be a smooth solution of Burger's equation. Assume that, for each $t \geq 0$, $u(\cdot, t)$ has compact support, and define its **mass** by

    $$M(t) = \int_{-\infty}^\infty u(x,t)\,dx\,.$$

    Prove that mass is conserved, i.e., $M(t)$ is constant in time. (*Hint:* Compute $\frac{d}{dt}M(t)$.)

    (c) Suppose u is a continuous integral solution of Burger's equation, i.e., u satisfies Eq. (2). Assume furthermore that $u(\cdot, t)$ has compact support for each $t \geq 0$. Show that mass is conserved also in this case. (*Hint:* Use test functions of the form $\phi(x,t) = a(x)b(t)$.)